Surds

N8 · Number · revision notes

Simplifying surds, calculating with them, expanding brackets and rationalising denominators, so that answers stay exact. Higher tier only: surds appear on their own on the non-calculator paper and inside Pythagoras, trigonometry and quadratic questions that ask for an exact answer.

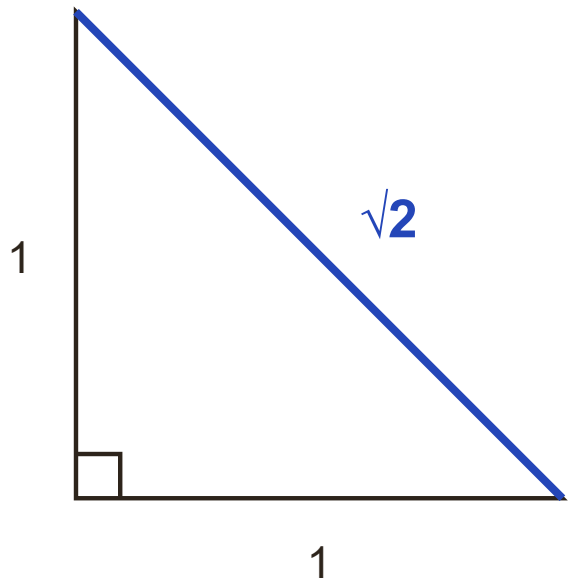
Key facts

- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
- $\sqrt{a} \times \sqrt{a} = a$
- $m\sqrt{a} + n\sqrt{a} = (m+n)\sqrt{a}$
- $\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$
- Grade 7+ only $(a+\sqrt{b})(a-\sqrt{b}) = a^2 - b$
- Square numbers to spot: 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144
- $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

Notes

What a surd is

- A **surd** is a root that cannot be written exactly as a whole number or a fraction, e.g. $\sqrt{2}$, $\sqrt{5}$ and $\sqrt[3]{7}$. But $\sqrt{9} = 3$, so $\sqrt{9}$ is not a surd.
- Surds are exact. Writing $\sqrt{2} \approx 1.414$ loses accuracy, so a question asking for an 'exact' answer wants a surd.



Pythagoras: 1² + 1² = 2, so the hypotenuse is exactly √2. The decimal 1.414 is only an approximation of it.

- A surd is in its simplest form when the number under the root has no square factor other than 1.

Simplifying and the rules

- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ and $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
- To simplify, find the **largest** square factor (4, 9, 16, 25, 36, 49, ...): $\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}$
- $\sqrt{a} \times \sqrt{a} = a$, so $(\sqrt{7})^2 = 7$ and $(2\sqrt{3})^2 = 4 \times 3 = 12$
- Add or subtract only like surds: $5\sqrt{3} + 2\sqrt{3} = 7\sqrt{3}$, but $\sqrt{2} + \sqrt{3}$ does not simplify. And $\sqrt{a+b}$ is not $\sqrt{a} + \sqrt{b}$
- Simplify each surd first to find like surds, for example:

$$\begin{aligned}\sqrt{12} + \sqrt{27} &= 2\sqrt{3} + 3\sqrt{3} \\ &= 5\sqrt{3}\end{aligned}$$

Expanding brackets Grade 7+ only

- Multiply every term in one bracket by every term in the other, as in algebra, then collect terms.

$$\begin{aligned}&(2+\sqrt{5})(4-\sqrt{5})\\&=8-2\sqrt{5}+4\sqrt{5}-5\\&=3+2\sqrt{5}\end{aligned}$$

- Squaring a bracket: write it out twice so you don't miss the middle term.

$$\begin{aligned}(3+\sqrt{2})^2&=(3+\sqrt{2})(3+\sqrt{2})\\&=9+3\sqrt{2}+3\sqrt{2}+2\\&=11+6\sqrt{2}\end{aligned}$$

- Difference of two squares: $(a+\sqrt{b})(a-\sqrt{b})=a^2-b$, with no surd left.

Rationalising the denominator Grade 7+ only

- To rationalise means to rewrite a fraction so there is no surd in the denominator.
- Single surd on the bottom: multiply top and bottom by that surd:

$$\begin{aligned}&\frac{10}{\sqrt{5}}\\&=\frac{10\sqrt{5}}{5}\\&=2\sqrt{5}\end{aligned}$$

- Bracket on the bottom: multiply top and bottom by the same bracket with the sign in the middle changed. For example: Grade 8+ only

$$\begin{aligned}&\frac{4}{3-\sqrt{5}}\\&=\frac{4(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}\\&=\frac{4(3+\sqrt{5})}{4}\\&=3+\sqrt{5}\end{aligned}$$

Grade 8+ only

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Simplify, or write in the form $k\sqrt{b}$

1 to 3 marks | **Grade 6**

1. List the square numbers: 4, 9, 16, 25, 36, 49, ...
2. Find the largest square number that is a factor.
3. Split the root and take the square root of that factor.
4. Collect like surds if you are adding or subtracting.

Example. (a) Write $\sqrt{75}$ in the form $k\sqrt{3}$, where k is an integer.
(b) Simplify $\sqrt{75} + \sqrt{12}$

Model answer.

(a) $\sqrt{25 \times 3} = 5\sqrt{3}$ (B1)

(b) $\sqrt{12} = 2\sqrt{3}$ (M1)

$5\sqrt{3} + 2\sqrt{3} = 7\sqrt{3}$ (A1)

Expand and simplify, giving the answer as $a + b\sqrt{c}$

2 marks | **Grade 7**

1. Write a squared bracket as two brackets.
2. Multiply every term by every term (4 terms).
3. Use $\sqrt{c} \times \sqrt{c} = c$, then collect the whole numbers and the surds.

Example. Expand and simplify $(5 - \sqrt{3})^2$
Give your answer in the form $a + b\sqrt{3}$, where a and b are integers.

Model answer.

$(5 - \sqrt{3})(5 - \sqrt{3})$

$= 25 - 5\sqrt{3} - 5\sqrt{3} + 3$ (M1 for at least 3 correct terms)

$= 28 - 10\sqrt{3}$ (A1)

Rationalise the denominator (single surd)

2 marks | Grade 7

1. Multiply the top and the bottom by the surd on the bottom.
2. Use $\sqrt{b} \times \sqrt{b} = b$ on the bottom.
3. Simplify the whole fraction.

Example. Rationalise the denominator of $\dfrac{14}{\sqrt{7}}$

Give your answer in its simplest form.

Model answer.

$$\frac{14}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{14\sqrt{7}}{7} \quad \text{(M1)}$$

$$= 2\sqrt{7} \quad \text{(A1)}$$

Show that: rationalise a denominator with a bracket

3 marks | Grade 8

1. Multiply the top and the bottom by the denominator with the middle sign changed.
2. Expand the bottom as a difference of two squares: the surds cancel.
3. Simplify and match the form given in the question.

Example. Show that $\dfrac{6}{\sqrt{5}-1}$ can be written as $\dfrac{3(\sqrt{5}+1)}{2}$

Model answer.

$$\frac{6}{\sqrt{5}-1} \times \frac{\sqrt{5}+1}{\sqrt{5}+1} \quad \text{(M1)}$$

Bottom: $(\sqrt{5}-1)(\sqrt{5}+1) = 5-1$

$$= 4 \quad \text{(M1)}$$

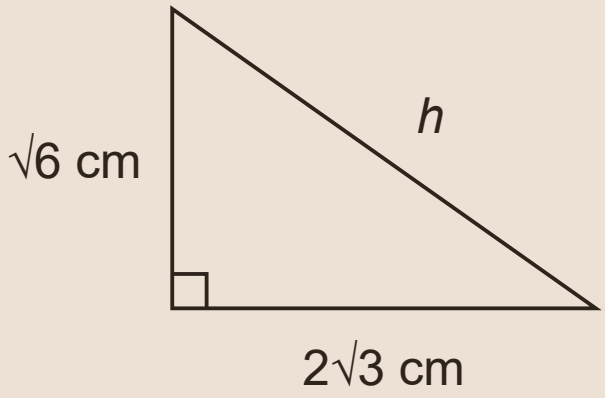
$$\frac{6(\sqrt{5}+1)}{4} = \frac{3(\sqrt{5}+1)}{2} \quad \text{(A1)}$$

Exact answer in a geometry problem

3 marks | Grade 8

1. Use the usual method (e.g. Pythagoras), keeping every value as a surd.
2. Square surds carefully: $(k\sqrt{b})^2 = k^2 b$
3. Simplify the final surd and give units.

Example. A right-angled triangle has shorter sides of length $2\sqrt{3}$ cm and $\sqrt{6}$ cm.
Work out the exact length of the hypotenuse. Give your answer as a surd in its simplest form.



Model answer.

$(2\sqrt{3})^2 = 12$ and $(\sqrt{6})^2 = 6$ (M1 for either)

$h^2 = 12 + 6 = 18$, so $h = \sqrt{18}$ (M1)

$h = 3\sqrt{2}$ cm (A1)

Don't lose marks

- Using a square factor that isn't the largest, e.g. $\sqrt{72} = 2\sqrt{18}$, and stopping. Keep going to $6\sqrt{2}$
- Adding unlike surds: $\sqrt{2} + \sqrt{3}$ is not $\sqrt{5}$
- Squaring a bracket term by term: $(5 - \sqrt{3})^2$ is not 25 + 3; the middle term $-10\sqrt{3}$ is needed.
- Multiplying only the denominator when rationalising: the numerator must be multiplied too.
- Giving a rounded decimal when the question asks for an exact answer.

More tips

Memory tricks

- Learn the square numbers up to 144 so you can spot the largest square factor quickly.
- Check a simplified surd: the number left under the root should have no square factor (4, 9, 16, ...).
- Treat a surd like a letter when collecting terms ($\sqrt{3}$ behaves like x), but remember $\sqrt{3} \times \sqrt{3} = 3$
- To rationalise $a + \sqrt{b}$ or $a - \sqrt{b}$ on the bottom, change the sign in the middle and multiply top and bottom by it.
- On the calculator paper, check by typing the original and your answer: both should give the same decimal.

Exam technique

- 'Exact' means leave your answer as a simplified surd (or in terms of π); don't round.
- 'In the form $a + b\sqrt{c}$' means match that form exactly, with integers for a and b
- In 'show that' questions, write the multiplying fraction and the expanded denominator on separate lines.
- Surds turn up in other topics: Pythagoras, trigonometry with exact values, and the quadratic formula with an exact answer.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade 6 — **Simplify a surd**
Take out the largest square factor: $\sqrt{48} = \sqrt{16} \times \sqrt{3} = 4\sqrt{3}$

Grade 6 — **Multiply and divide surds**
$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, e.g. $\sqrt{6} \times \sqrt{10} = \sqrt{60} = 2\sqrt{15}$, and $\sqrt{20} \div \sqrt{5} = \sqrt{4} = 2$

Grade 6 — **Add and subtract surds**
Simplify first, then collect like surds: $\sqrt{50} + \sqrt{8} = 5\sqrt{2} + 2\sqrt{2} = 7\sqrt{2}$

Grade 7 — **Expand brackets containing surds**
Multiply every term by every term: $(3 + \sqrt{2})(1 - \sqrt{2}) = 1 - 2\sqrt{2}$

Grade 7 — **Rationalise a denominator with a single surd**
For $\dfrac{6}{\sqrt{3}}$, multiply top and bottom by $\sqrt{3}$: $\dfrac{6\sqrt{3}}{3} = 2\sqrt{3}$

Grade 8 — **Rationalise a denominator containing a bracket**
For $\dfrac{1}{3 + \sqrt{2}}$, multiply top and bottom by $3 - \sqrt{2}$, since $(3 + \sqrt{2})(3 - \sqrt{2}) = 7$, giving $\dfrac{3 - \sqrt{2}}{7}$

Grade 8 — **Give exact answers in geometry problems**
Keep surds in Pythagoras and trigonometry, e.g. a hypotenuse of $\sqrt{72} = 6\sqrt{2}$ cm.

Grade 9 — **Solve multi-step surd problems**
E.g. find integers a and b when an area or expression is written as $a + b\sqrt{c}$

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