Recurring decimals

N10 · Number · revision notes

Writing fractions as recurring decimals, and changing recurring decimals into fractions using algebra. This is Higher tier only, and usually appears as a 2 to 3 mark 'show that' or 'prove algebraically' question on the non-calculator paper.

Key facts

- A fraction in its simplest form terminates only if its denominator has no prime factors other than 2 and 5
- $0.\dot{7} = \frac{7}{9}$, $0.\dot{1}\dot{8} = \frac{18}{99}$, $0.\dot{1}2\dot{3} = \frac{123}{999}$
- Subtract the two equations so the recurring tails cancel, then simplify the fraction
- 1 repeating digit: × 10; 2 repeating digits: × 100; 3 repeating digits: × 1000
- $\frac{1}{3} = 0.\dot{3}$, $\frac{2}{3} = 0.\dot{6}$, $\frac{1}{9} = 0.\dot{1}$, $\frac{1}{11} = 0.\dot{0}\dot{9}$, $\frac{1}{6} = 0.1\dot{6}$

Notes

Terminating and recurring decimals

- A **terminating** decimal stops, e.g. $\frac{3}{8} = 0.375$. A **recurring** decimal has a digit or block of digits that repeats for ever, e.g. $\frac{5}{11} = 0.454545...$
- Every fraction gives a terminating or a recurring decimal. In short division, once a remainder repeats, the digits repeat.

$$
\begin{array}{r}
0.4\,5\,4\,5\ldots \\
11\overline{)\,5.^{5}0\,^{6}0\,^{5}0\,^{6}0}
\end{array}
$$

remainders 5, 6, 5, 6, ...

$\frac{5}{11}$ = 5 ÷ 11: the remainders 5, 6, 5, 6 repeat, so the digits 4, 5 repeat for ever.

- A fraction in its simplest form terminates only if the prime factors of its denominator are 2s and 5s: $\frac{7}{40}$ terminates ($40 = 2^3 \times 5$), but $\frac{7}{30}$ recurs (30 has a factor 3).
- Dot notation: one dot means that single digit repeats: $0.\dot{6} = 0.666...$ and $0.2\dot{3} = 0.2333...$
- Two dots mark the first and last digits of the repeating block: $0.\dot{4}\dot{5} = 0.4545...$ and $0.\dot{1}4\dot{2} = 0.142142...$

Recurring decimal to fraction

- Let x = the decimal, and write out a few repeats.
- Multiply by 10 if one digit repeats, by 100 if two digits repeat, by 1000 if three digits repeat.
- Subtract so that the recurring parts cancel exactly, then solve for x and simplify.
- Example: $x = 0.363636...$, so $100x = 36.363636...$ Subtracting, $99x = 36$, so

$$x = \frac{36}{99} = \frac{4}{11}$$

	100x =	36.363636...	same digits, so they cancel
−	x =	0.363636...	
	99x =	**36**	

Line up the decimal points: the recurring parts are identical, so subtracting leaves a whole number.

When the repeating starts later Grade 7+ only

- Multiply to get the non-recurring digits in front of the decimal point, then multiply again to shift one more full block. Subtract these two.
- Example: $x = 0.1666...$, so $10x = 1.666...$ and $100x = 16.666...$ Subtracting, $90x = 15$, so

$$x = \frac{15}{90} = \frac{1}{6}$$

- Two-digit block: $x = 0.2454545...$, so $10x = 2.454545...$ and $1000x = 245.454545...$ Subtracting, $990x = 243$, so

$$x = \frac{243}{990} = \frac{27}{110}$$

- Check before subtracting: the two numbers must have exactly the same digits after the decimal point.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Write a fraction as a recurring decimal

2 marks | Grade 5

1. Use short division: numerator ÷ denominator, adding zeros after the decimal point.
2. Stop when a remainder repeats: from there the digits repeat.
3. Write the answer with dots over the repeating digit or block.

Example. Write $\frac{7}{12}$ as a recurring decimal.

Model answer.

7.000 ÷ 12: 70 ÷ 12 = 5 r 10, 100 ÷ 12 = 8 r 4, 40 ÷ 12 = 3 r 4, ... (M1 for a correct division showing a repeating remainder)

$0.58\dot{3}$ (A1)

Prove algebraically that a recurring decimal equals a fraction

2 marks | Grade 6

1. Write x = the decimal with at least two repeats shown.
2. Multiply by the power of 10 that shifts one whole block.
3. Subtract, solve for x and simplify to the given fraction.

Example. Prove algebraically that $0.\dot{2}\dot{7} = \frac{3}{11}$

Model answer.

Let $x = 0.272727\ldots$, so $100x = 27.272727\ldots$ (M1 for both, with an intention to subtract)

$100x - x = 99x = 27$, so $x = \frac{27}{99} = \frac{3}{11}$ (A1 for a complete proof)

Recurring decimal with a non-recurring part to a fraction

2 to 3 marks | Grade 7

1. Let x = the decimal.
2. Find two multiples of x (such as $10x$ and $100x$) whose digits after the point are identical.
3. Subtract, solve and simplify.

Example. Write $0.3\dot{8}$ as a fraction in its simplest form.

Model answer.

$x = 0.3888\ldots,\ 10x = 3.888\ldots,\ 100x = 38.888\ldots$ (M1)

$100x - 10x = 90x = 35$ (M1)

$$x = \frac{35}{90} = \frac{7}{18}$$ (A1)

Explain whether a fraction is terminating or recurring

1 to 2 marks | Grade 7

1. Simplify the fraction first.
2. Write the denominator as a product of primes.
3. Only 2s and 5s: it terminates. Any other prime factor: it recurs.

Example. Without dividing, explain whether $\frac{9}{75}$ can be written as a terminating decimal.

Model answer.

$\frac{9}{75} = \frac{3}{25}$ in its simplest form (M1)

$25 = 5^2$ has no prime factors other than 2 and 5, so it is a terminating decimal (A1)

(In fact $\frac{3}{25} = 0.12$.)

Don't lose marks

- Multiplying by 10 when two digits repeat: the tails don't line up, so they don't cancel.
- Writing $x = 0.27$ without the repeats or the '...', which is not a recurring decimal.
- Forgetting to simplify: $\frac{27}{99}$ must become $\frac{3}{11}$
- Deciding a fraction recurs before simplifying it: $\frac{9}{75} = \frac{3}{25}$ terminates.
- Misreading the dots: $0.2\dot{3}$ is 0.2333..., but $0.\dot{2}\dot{3}$ is 0.232323...

More tips

Memory tricks

- Count the digits in the repeating block: that is how many zeros go on the multiplier.
- Check: a decimal that recurs from the first digit is the repeating block over the same number of 9s, e.g. $0.\dot{2}\dot{7} = \frac{27}{99}$. Use this to check, but show algebra in a 'prove' question.
- Write the two decimals one under the other before subtracting, so you can see the tails match.
- On the calculator paper, check your fraction by dividing the numerator by the denominator.

Exam technique

- 'Prove algebraically' means start with $x = \ldots$ and show the subtraction. Dividing 3 by 11 does not prove the result.
- Write at least two or three repeats (e.g. 0.272727...) so the cancelling is clear to the examiner.
- End with a clear conclusion line that matches the fraction you were asked to show.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade 5 — **Write a fraction as a recurring decimal**

Use short division until the remainders repeat, e.g. $\frac{5}{11} = 0.454545\ldots$

Grade 5 — **Use dot notation for recurring decimals**

Dots go over the first and last digits of the repeating block: $0.\dot{1}2\dot{3} = 0.123123\ldots$

Grade 6 — **Tell whether a fraction terminates or recurs**

In its simplest form, a fraction terminates only if the denominator has no prime factors other than 2 and 5.

Grade 6 — **Convert a pure recurring decimal to a fraction**

E.g. $x = 0.\dot{7}\dot{2}$: $100x - x = 72$, so

$$x = \frac{72}{99} = \frac{8}{11}$$

Grade 7 — **Convert decimals with a non-recurring part**

E.g. $x = 0.4\dot{1}$:

$$\begin{aligned} &100x - 10x \\ &= 41.11\ldots - 4.11\ldots \\ &= 37 \end{aligned}$$

so $x = \frac{37}{90}$

Grade 7 — **Prove a recurring decimal equals a fraction**

Show the full algebraic method, from 'let x = ...' to the simplified fraction.

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