Algebraic proof

A6 · Algebra · revision notes

Higher only. Using algebra to prove that a statement is always true, for example about odd, even and consecutive numbers, and showing that two expressions are identical. Proofs are usually 3 to 4 marks and need a clear final statement.

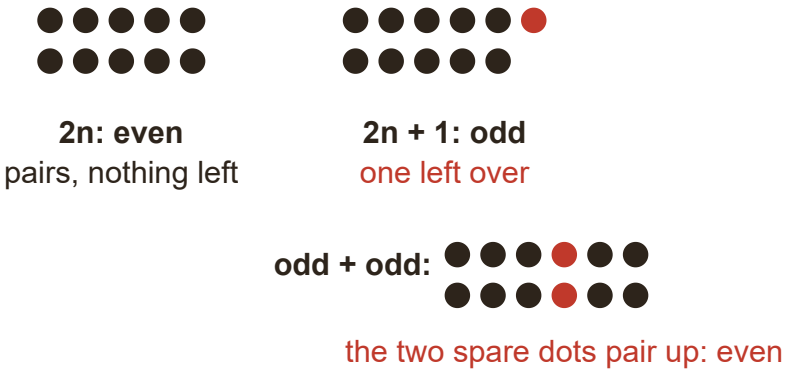
Key facts

- Even: $2n$. Odd: $2n + 1$
- Consecutive odd numbers: $2n + 1$, $2n + 3$
- odd + odd = even; even + even = even; odd × odd = odd
- $\equiv$: identical for every value
- Consecutive integers: n, $n + 1$, $n + 2$
- Multiple of k: k × an integer
- Counterexample: one example that shows a statement is false

Notes

Building blocks

- Let n be an integer. An even number is $2n$ and an odd number is $2n + 1$ (or $2n - 1$).



Why $2n$ is even and $2n + 1$ is odd, and why odd + odd is even.

- Consecutive integers: n, $n + 1$, $n + 2$. Consecutive even numbers: $2n$, $2n + 2$. Consecutive odd numbers: $2n + 1$, $2n + 3$
- Two odd numbers that are not related need different letters: $2m + 1$ and $2n + 1$
- A multiple of k can be written as k × an integer, e.g. $6(n^2 + 1)$ is a multiple of 6 when n is an integer.

Writing a proof

- Write the numbers in terms of n, then expand and simplify.
- Factorise to show the form you need: $2(\ldots)$ for even, $2(\ldots)+1$ for odd, $k(\ldots)$ for a multiple of k.
- Finish with a sentence linking back to the statement, e.g. '$2(m+n+1)$ is 2 × an integer, so the sum is even.'
- Examples with numbers are **not** a proof, however many you try. But one counterexample is enough to **disprove** a statement.

Identities

- An **equation** is true only for particular values: $2x+1=7$ only when $x=3$
- An **identity** is true for every value of the letter and is written with $\equiv$, e.g. $2(x+3)\equiv 2x+6$
- To show $A\equiv B$, expand and simplify one side until it matches the other.
- Show $(n+3)^2-(n+1)^2\equiv 4(n+2)$:

$$\begin{aligned} n^2+6n+9-(n^2+2n+1) &= 4n+8 \\ &= 4(n+2) \end{aligned}$$

- This also proves that $(n+3)^2-(n+1)^2$ is always a multiple of 4 when n is an integer.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, C = a communication mark.

Prove a statement about consecutive or even numbers 3 marks **Grade 7**

1. Write the numbers in terms of n.
2. Add (or multiply) and simplify.
3. Factorise and write a conclusion.

Example. Prove that the sum of any three consecutive even numbers is a multiple of 6.

Model answer.

$2n+(2n+2)+(2n+4)$ M1

$=6n+6$ M1

$=6(n+1)$, which is 6 × an integer, so it is a multiple of 6 C1

Prove a result using expanding

4 marks | Grade 8

1. Write the numbers in algebra, e.g. consecutive odd numbers $2n+1$ and $2n+3$
2. Expand each bracket carefully, putting the subtracted one in brackets.
3. Simplify, factorise and write a conclusion.

Example. Prove that the difference between the squares of any two consecutive odd numbers is a multiple of 8.

Model answer.

$(2n+3)^2-(2n+1)^2$ M1

$=4n^2+12n+9-(4n^2+4n+1)$ M1

$=8n+8$ A1

$=8(n+1)$, which is a multiple of 8 C1

Show that an identity is true

2 marks | Grade 6

1. Start with the more complicated side.
2. Expand and simplify until it matches the other side.

Example. Show that $(x+5)^2-(x-5)^2 \equiv 20x$

Model answer.

$x^2+10x+25-(x^2-10x+25)$ M1

$=20x$ A1

Show that a statement is false

2 marks | Grade 6

1. Try values until one fails.
2. Show the working for that value and say clearly why it disproves the statement.

Example. Sam says, 'For every positive integer n, n^2+n+11 is a prime number.' Show that Sam is wrong.

Model answer.

e.g. $n=10$: $100+10+11=121$

$=11\times 11$ M1

121 is not prime, so Sam is wrong C1

Don’t lose marks

- Using examples with numbers as a 'proof'.
- Using n and $n + 1$ for two odd numbers: these are consecutive integers, and one of them is even.
- Using the same letter for two unrelated odd numbers, which only proves the result when they are equal.
- Stopping at $8n + 8$ without factorising and writing a conclusion.

More tips

Memory tricks

- Test the statement with a couple of numbers first to see why it is true, then write the algebra.
- Aim for the form you need: factorise out the number in the statement (2, 4, 6, 8, …).
- Use different letters for numbers that are not linked.

Exam technique

- End every proof with a statement, e.g. '= 8(n + 1), which is a multiple of 8.'
- Say what your letters stand for: 'Let n be an integer.'
- Don't start by assuming the result is true: start from one side and show it becomes the other.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade 6 — **Write expressions for odd, even and consecutive numbers**
Even $2n$, odd $2n + 1$, consecutive integers n, $n + 1$, $n + 2$

Grade 6 — **Disprove a statement with a counterexample**
Find one example where the statement is false and show why it is false.

Grade 6 — **Show that two expressions are identical**
Expand and simplify one side until it matches the other exactly.

Grade 7 — **Prove results about odd and even numbers**
e.g. the sum of two odd numbers: $(2m + 1) + (2n + 1) = 2(m + n + 1)$, which is even.

Grade 7 — **Prove an expression is a multiple**
Factorise to show it is k × an integer, e.g. $6n + 6 = 6(n + 1)$

Grade 8 — **Prove results about consecutive and square numbers**
e.g. the difference between the squares of two consecutive odd numbers is a multiple of 8.

Grade 9 — **Construct harder multi-step proofs**
Rearrange into a useful form, e.g. $(2n + 1)(2n + 3) = 4(n^2 + 2n + 1) - 1$, one less than a multiple of 4.

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