Iteration

A20 · Algebra · revision notes

Higher only. Finding approximate solutions of equations that cannot be solved exactly: showing that a solution lies between two values by a change of sign, rearranging an equation into an iterative formula, and using $x_{n+1} = \ldots$ to find a solution.

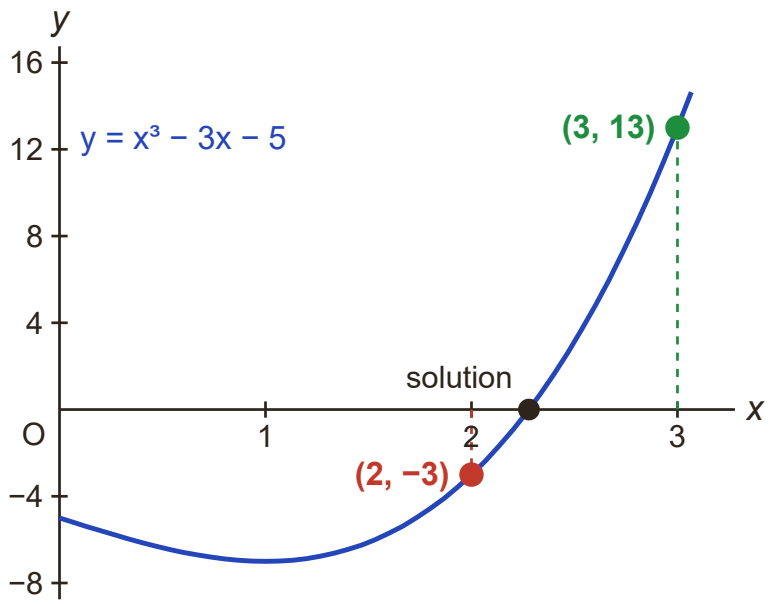
Key facts

- Change of sign between $x = a$ and $x = b$ means a solution between a and b
- x_1 comes from x_0, x_2 from x_1, and so on
- Converging values approach a solution of the original equation
- Iterative formula: $x_{n+1} = g(x_n)$, from rearranging to $x = g(x)$
- Use ANS on the calculator to keep full accuracy

Notes

Showing a solution lies in an interval

- Rearrange the equation so that one side is 0.
- Substitute the two end values. If one result is positive and the other is negative, there is a change of sign, so the graph crosses the x-axis between them: there is a solution in that interval.
- $x^3 - 3x - 5 = 0$ between 2 and 3: when $x = 2$, $8 - 6 - 5 = -3$; when $x = 3$, $27 - 9 - 5 = 13$
- Finish with a conclusion: 'There is a change of sign, so there is a solution between 2 and 3.'



Below the axis at $x = 2$, above it at $x = 3$: the curve must cross in between.

Rearranging into an iterative formula

- Rearrange the equation into the form $x = \ldots$ (with x also on the right), then write it as $x_{n+1} = \ldots$ with x_n on the right.
- $x^3 - 3x - 5 = 0$ gives $x^3 = 3x + 5$, so $x = \sqrt[3]{3x + 5}$ and the formula is $x_{n+1} = \sqrt[3]{3x_n + 5}$
- In 'show that' questions, work from the equation to the given form, one step per line.

Iterating

- Start with the given starting value x_0 (or x_1). Substitute it to get the next value, then repeat.
- Calculator: type the starting value and press =. Then type the formula with ANS in place of x_n, and press = once for each iteration.
- $x_{n+1} = \sqrt[3]{3x_n + 5}$ with $x_0 = 2$ gives $x_1 = 2.22398$, $x_2 = 2.26837$, $x_3 = 2.27697$, …
- The values get closer and closer to (converge to) a solution of the original equation, here 2.279 to 3 decimal places.
- Write each value to at least 4 or 5 decimal places while you work.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Show that there is a solution between two values 2 marks **Grade 6**

1. Make sure the equation is in the form ... = 0.
2. Substitute both values and write down the results.
3. State that there is a change of sign, so there is a solution between them.

Example. Show that the equation $x^3 + 2x - 7 = 0$ has a solution between $x = 1$ and $x = 2$

Model answer.

$x = 1$: $1 + 2 - 7 = -4$; $x = 2$: $8 + 4 - 7 = 5$ M1

There is a change of sign, so there is a solution between 1 and 2 A1

Show that the equation can be rearranged 2 marks **Grade 7**

1. Start from the given equation.
2. Rearrange one step at a time until you reach the given form exactly.

Example. Show that $x^3 + 2x - 7 = 0$ can be rearranged to give $x = \sqrt[3]{7 - 2x}$

Model answer.

$x^3 = 7 - 2x$ M1

$x = \sqrt[3]{7 - 2x}$ A1

Use the iteration formula

3 marks | Grade 7

1. Substitute the starting value to find x_1
2. Use each answer to find the next one (use ANS).
3. Write down every value, then give the final estimate.

Example. Starting with $x_0 = 1.5$, use the iteration formula $x_{n+1} = \sqrt[3]{7 - 2x_n}$ three times to find an estimate for a solution of $x^3 + 2x - 7 = 0$

Model answer.

$x_1 = \sqrt[3]{7 - 2 \times 1.5} = 1.58740$ M1

$x_2 = 1.56393$ M1

$x_3 = 1.57030$ A1

Explain what the result represents

1 mark | Grade 8

1. Link the converging value back to the original equation.
2. Use the word 'solution' (or 'root') and name the equation.

Example. The values from $x_{n+1} = \sqrt[3]{7 - 2x_n}$ get closer and closer to 1.569. What does 1.569 represent?

Model answer. An estimate of a solution of the equation $x^3 + 2x - 7 = 0$ B1

Don't lose marks

- Substituting the wrong value: x_2 comes from x_1, not from x_0
- Rounding each value heavily before using it, which makes later values wrong. Use ANS instead.
- Not writing a conclusion after the change-of-sign check.
- Typing a cube root over only part of the expression: use brackets.

More tips

Memory tricks

- Enter the starting value and press =; then every press of = after typing the formula with ANS gives the next iteration.
- Write each value to at least 5 decimal places: examiners check each one.
- If the values jump about wildly or grow quickly, check how you typed the formula.

Exam technique

- Show every iteration, not just the last one: the method marks are for x_1, x_2, …
- 'Show that' a solution lies in an interval needs both substituted values and the words 'change of sign'.
- Give your final answer to the accuracy asked for.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade 6

Show a solution lies between two values

Substitute both values into the equation written as ... = 0; a change of sign shows there is a solution between them.

Grade 7

Use an iterative formula

Substitute x_0 to get x_1, then x_1 to get x_2, and so on, using ANS on your calculator.

Grade 7

Find a solution to a given accuracy

Keep iterating until successive values round to the same answer, e.g. to 3 decimal places.

Grade 7

Show an equation rearranges to a given form

e.g. show that $x^3 + 4x - 1 = 0$ can be written as $x = \dfrac{1 - x^3}{4}$

Grade 8

Explain what the iterated values represent

The values get closer to a solution of the original equation.

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