Completing the square

A11, A18 · Algebra · revision notes

Higher only. Writing a quadratic in the form $(x+p)^2+q$ or $a(x+p)^2+q$, and using it to solve equations with exact (surd) answers and to find the turning point of the graph.

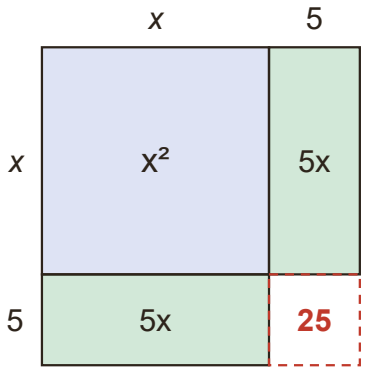
Key facts

- $x^2+bx+c=\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c$
- $y=(x+p)^2+q$ has turning point $(-p,q)$
- $(x+p)^2=k$ gives $x=-p\pm\sqrt{k}$
- Line of symmetry: $x=-p$
- Grade 8+ only
  $ax^2+bx+c=a\left(x^2+\frac{b}{a}x\right)+c$, then complete the square inside
- A square is never negative: $(x+p)^2\geq 0$

Notes

The method when a = 1

- Halve the coefficient of x to get p. Then $x^2 + bx + c = (x + p)^2 - p^2 + c$
- The $-p^2$ removes the extra p^2 you get when $(x + p)^2$ is expanded.



x² + 10x
= (x + 5)² − 25
(the square is 25 too big)

$x^2 + 10x$ fills the square $(x + 5)^2$ except the red corner, so subtract 25.

$$x^2 + 10x + 7 = (x + 5)^2 - 25 + 7$$
$$= (x + 5)^2 - 18$$

- Odd coefficients give fractions, for example:

$$x^2 - 3x + 1$$
$$= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 1$$
$$= \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$$

When a is not 1 Grade 8+ only

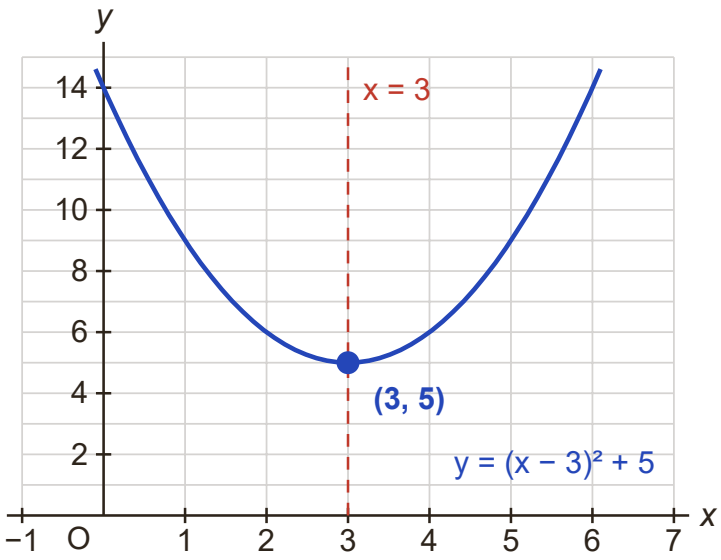
- Take a out as a factor of the x^2 and x terms only: $3x^2 + 12x - 5 = 3(x^2 + 4x) - 5$
- Complete the square inside the bracket, then multiply back out:
  $3[(x + 2)^2 - 4] - 5 = 3(x + 2)^2 - 17$

Solving equations

- Complete the square, rearrange to $(x + p)^2 = k$, then square root both sides, remembering ±.
- $x^2 - 4x - 3 = 0$: $(x - 2)^2 - 7 = 0$, so $(x - 2)^2 = 7$ and $x = 2 \pm \sqrt{7}$
- This gives exact answers in surd form, which is why questions ask for answers 'in the form $a \pm \sqrt{b}$'.

Turning points

- For $y = (x + p)^2 + q$, the minimum point is $(-p, q)$: the squared bracket is never negative, so the smallest value of y is q, when $x = -p$
- $y = (x - 3)^2 + 5$ has turning point $(3, 5)$: the x-coordinate changes sign, the y-coordinate does not.



Turning point $(3, 5)$; line of symmetry $x = 3$

- The line of symmetry of the graph is $x = -p$
- For $y = q - (x + p)^2$, the turning point $(-p, q)$ is a maximum. Grade 8+ only
- Because $(x + p)^2 \geq 0$, if q is positive then $(x + p)^2 + q$ is always positive. Use this in 'prove that' questions. Grade 8+ only

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, C = a communication mark.

Write in the form $(x + a)^2 + b$

2 marks | Grade 6

1. Halve the coefficient of x to find a.
2. Subtract a^2 and add the constant.

Example. Write $x^2 - 8x + 21$ in the form $(x + a)^2 + b$

Model answer.

$(x - 4)^2 - 16 + 21$ M1

$(x - 4)^2 + 5$ A1

Solve by completing the square (surd answers)

3 marks | Grade 7

1. Complete the square.
2. Rearrange to (bracket)^2 = number.
3. Square root both sides with ±, then rearrange.

Example. Solve $x^2 + 6x - 2 = 0$ by completing the square. Give your answers in the form $a \pm \sqrt{b}$

Model answer.

$(x+3)^2 - 9 - 2 = 0$, so $(x+3)^2 = 11$ M1

$x + 3 = \pm\sqrt{11}$ M1

$x = -3 \pm \sqrt{11}$ A1

Hence find the turning point

3 marks | Grade 7

1. Complete the square.
2. Read off $(-p, q)$ from $(x+p)^2 + q$

Example. (a) Write $x^2 + 10x + 32$ in the form $(x+p)^2 + q$.
(b) Hence write down the coordinates of the turning point of $y = x^2 + 10x + 32$

Model answer.

(a) $(x+5)^2 - 25 + 32$ M1, $(x+5)^2 + 7$ A1

(b) $(-5, 7)$ B1

(follow through from (a))

Complete the square when a is not 1

3 marks | Grade 8

1. Factor a out of the x^2 and x terms.
2. Complete the square inside the bracket.
3. Multiply out the square bracket and collect the numbers.

Example. Write $2x^2 + 12x + 7$ in the form $a(x+b)^2 + c$

Model answer.

$2(x^2 + 6x) + 7$ M1

$2[(x+3)^2 - 9] + 7$ M1

$2(x+3)^2 - 11$ A1

Prove that a quadratic is always positive

3 marks | Grade 8

1. Complete the square.
2. State that the square is greater than or equal to 0 for all x.
3. Conclude that the whole expression is at least q, which is positive.

Example. Prove that $x^2 - 6x + 10 > 0$ for all values of x.

Model answer.

$x^2 - 6x + 10 = (x - 3)^2 - 9 + 10$ M1

$= (x - 3)^2 + 1$ A1

$(x - 3)^2 \geq 0$ for all x, so $(x - 3)^2 + 1 \geq 1 > 0$ C1

Don't lose marks

- Forgetting to subtract p^2: $x^2 + 6x + 1$ is not $(x + 3)^2 + 1$
- Getting the sign of the turning point wrong: $(x - 4)^2 + 5$ has turning point $(4, 5)$, not $(-4, 5)$
- Missing the ± when square rooting, so only one solution is found.
- When a is not 1, forgetting to multiply the $-p^2$ by a.

More tips

Memory tricks

- Halve it, square it, take it away: halve b to get p, then subtract p^2
- Check by expanding: $(x - 4)^2 + 5 = x^2 - 8x + 21$, which matches.
- Turning point of $(x + p)^2 + q$: change the sign of p, keep the sign of q.

Exam technique

- If the question says 'by completing the square', using the formula instead will not get the marks.
- 'Hence' means use your completed square form from the part before.
- Leave answers as surds when the question asks for exact answers or a form like $a \pm \sqrt{b}$

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade 6 **Complete the square for simple quadratics**

Halve the coefficient of x:

$$x^2 + 6x + 1 = (x + 3)^2 - 9 + 1$$
$$= (x + 3)^2 - 8$$

Grade 7 **Solve a quadratic by completing the square**

Rearrange to $(x+p)^2=k$ and square root both sides, e.g. $(x+3)^2=8$ gives $x=-3\pm2\sqrt{2}$

Grade 7

Find the turning point from completed square form

$y=(x+p)^2+q$ has its minimum point at $(-p,q)$

Grade 7

Find unknowns by comparing forms

Expand the completed square and match coefficients, e.g. $x^2+bx+11\equiv(x+4)^2+c$ gives $b=8$, $c=-5$

Grade 8

Complete the square when a is not 1

Take out a factor of a first, e.g. $2x^2-8x+3=2(x-2)^2-5$

Grade 8

Prove a quadratic is always positive

Complete the square and use the fact that a square is never negative.

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