Numerical methods in context

P9.4 · Pure mathematics · revision notes

Many equations and integrals that come from real situations cannot be solved exactly. You apply change of sign, iteration, Newton–Raphson and the trapezium rule to models and data, then interpret the answer in context, with units and a sensible accuracy, and comment on how reliable it is. These questions often combine several topics and are worth 6 to 10 marks.

Key facts

- Change of sign: f continuous, $\mathrm{f}(a)$ and $\mathrm{f}(b)$ of opposite signs $\Rightarrow$ a root between a and b
- Newton–Raphson: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$
- Distance $= \int v \, \mathrm{d}t$ (area under a speed–time graph)
- Increasing at a decreasing rate (concave): the trapezium rule underestimates
- Iteration: $x_{n+1} = \mathrm{g}(x_n)$
- Trapezium rule: $\dfrac{1}{2}h\{(y_0 + y_n) + 2(y_1 + \cdots + y_{n-1})\}$
- Decimal hours to minutes: multiply the decimal part by 60

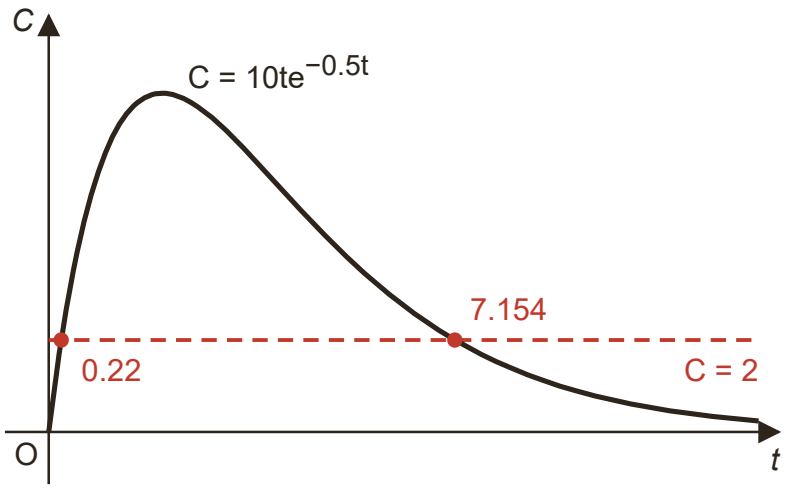
Notes

Why numerical methods

- Equations such as $t = 2\ln(5t)$ or $\theta - \sin\theta = 1$ have no algebraic solution, so you find an approximate root.
- Data given only as a table of values cannot be integrated exactly, so you use the trapezium rule.

Setting up and solving

- Write the condition as an equation, then rearrange it into the form the question asks for ($\mathbf{f}(x) = 0$ or $x = \mathbf{g}(x)$).
- Choose a starting value that makes sense in the context, e.g. after the maximum if you want the time when a quantity falls below a level.
- A model may give two solutions: decide which one answers the question.

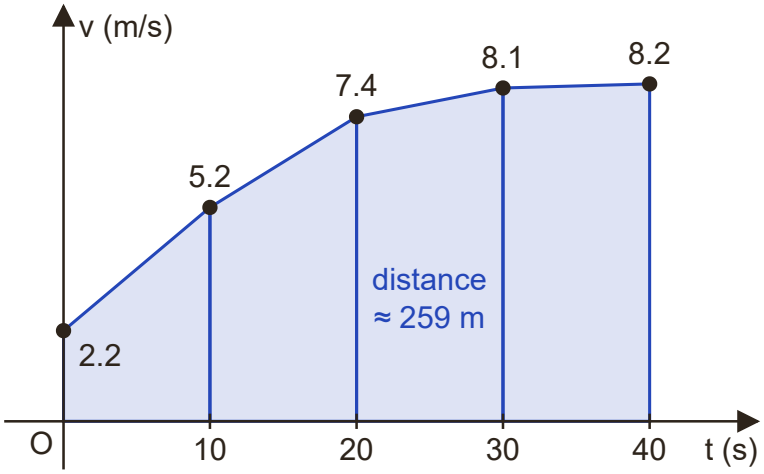


$C = 2$ twice: early on as the drug builds up, and at $t = 7.154$ as it wears off. Only the second answers 'when does it stop being effective?'

- Optimisation: setting a derivative equal to zero can give an equation that you then solve numerically.

Integrals in context

- The area under a rate graph is a total amount: under a speed–time graph it is distance; under a flow-rate graph, volume.
- Units multiply: (units of y) × (units of x), e.g. m s^{-1} × s = m.



The area under a speed–time graph is the distance. The real curve bends downwards between the readings, so the trapezia slightly underestimate it.

Interpreting and evaluating

- State the answer in the context of the question, with units and a sensible accuracy (e.g. to the nearest minute).
- Trapezium rule: say whether the estimate is too high or too low from the shape of the graph, and that more frequent readings (smaller h) would improve it.
- Question the model: is the answer within the range where the model is valid? Are the data accurate?

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, A1* = the solution must be fully correct, with every step shown.

Estimate a total from data with the trapezium rule 4 marks **Grade C**

1. Identify what the area under the graph represents (e.g. distance under a speed–time graph).
2. Find h from the spacing of the readings and apply the rule to all the values.
3. Give units, and use the shape of the data to say whether it is an over- or underestimate.

Example. A cyclist's speed, v m s^{-1}, is recorded every 10 seconds.

t (s)	0	10	20	30	40
v (m s^{-1})	2.2	5.2	7.4	8.1	8.2

(a) Use the trapezium rule with all the values to estimate the distance travelled in the 40 seconds.
(b) The speed increases at a decreasing rate. State, with a reason, whether your answer to (a) is an overestimate or an underestimate.

Model answer.

(a) $h = 10$ **B1**

$\frac{1}{2} \times 10 \times [2.2 + 8.2 + 2(5.2 + 7.4 + 8.1)]$

$= 5 \times 51.8$ **M1**

$= 259$ m **A1**

(b) Underestimate: the speed–time graph is concave (bends downwards), so the tops of the trapezia lie below the curve **B1**

Solve a model equation numerically and interpret the answer

8 marks | Grade B

1. Use a change of sign to show when the event happens.
2. Rearrange the condition into the given iterative form.
3. Iterate with ANS, then interpret the root in context with units.

Example. The concentration of a drug in a patient's blood, C mg per litre, t hours after an injection, is modelled by $C = 10te^{-0.5t}$. The drug is effective while $C \geq 2$.
(a) Show that the concentration falls to 2 mg per litre between $t = 7$ and $t = 7.5$.
(b) Show that when $C = 2$, $t = 2\ln(5t)$.
(c) Use the iteration $t_{n+1} = 2\ln(5t_n)$ with $t_0 = 7$ to find t_1, t_2 and t_3, to 3 decimal places.
(d) The root is 7.154 to 3 decimal places. Find, to the nearest minute, how long after the injection the drug stops being effective.

Model answer.

(a) $C(7) = 2.11\ldots > 2$ and $C(7.5) = 1.76\ldots < 2$; C is continuous, so $C = 2$ between these times — **M1 A1**

(b) $10te^{-0.5t} = 2 \Rightarrow e^{0.5t} = 5t \Rightarrow 0.5t = \ln(5t)$ **M1** $\Rightarrow t = 2\ln(5t)$ — **A1***

(c) $t_1 = 7.111$, $t_2 = 7.142$, $t_3 = 7.151$ — **M1 A1**

(d) $0.154 \times 60 = 9.2$ minutes, so about 7 hours 9 minutes after the injection — **M1 A1**

Comment on the reliability of a numerical answer

2 marks | Grade A

1. Check whether the answer lies inside the range of the data used to build the model.
2. Think about what the model ignores and how accurate the data are.
3. Make each comment specific to the context.

Example. The model $C = 10te^{-0.5t}$ for the concentration of a drug was fitted to readings taken during the first 5 hours after an injection. It predicts that the drug stops being effective about 7 hours 9 minutes after the injection.
Give two reasons why this prediction may not be reliable.

Model answer.

The prediction is outside the range of the data (after 5 hours), so the model may not hold at that time — **B1**

The model does not allow for differences between patients (e.g. how quickly their bodies remove the drug), and the readings it was fitted to may contain measurement errors — **B1**

Don't lose marks

- ✗ Giving a bare number with no units or context.
- ✗ Converting 7.154 hours to '7 hours 15 minutes' (0.154 hours is about 9 minutes).

- Finding the wrong root of a model that has two solutions.
- Using the number of readings instead of the number of intervals to find h
- Generic comments such as 'the data might be wrong': tie reliability comments to the specific model and data.

More tips

Memory tricks

- Rate × time = amount: the trapezium rule on a table of rates gives a total amount.
- Decimal hours to minutes: multiply the decimal part by 60.
- Before iterating, check that the starting value is on the correct side of any maximum or minimum for the root you want.
- Sense check: does the answer fit the story (a positive time, a realistic size)?

Exam technique

- Read what the answer represents (a time of day, a duration, a distance) and answer exactly the question asked.
- Give values to the accuracy requested, and carry full accuracy in the calculator between parts.
- For 'evaluate' or 'comment' marks, refer to the numbers in the question and say whether an estimate is too big or too small, and why.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade D — **Form an equation from a context**
Turn a condition such as 'the concentration falls to 2 mg per litre' into an equation $\mathrm{f}(t) = 0$

Grade C — **Estimate totals from data with the trapezium rule**
Distance $= \int v \, \mathrm{d}t$, volume $= \int (\text{flow rate}) \, \mathrm{d}t$, cross-section area $= \int (\text{depth}) \, \mathrm{d}x$

Grade C — **Solve contextual equations numerically**
Use a given iteration or Newton–Raphson, and a change of sign to locate or confirm the root.

Grade B — **Interpret numerical answers in context**
Give units and convert where sensible, e.g. 7.154 hours is about 7 hours 9 minutes.

Grade A — **Judge the reliability of an estimate**
Consider over- or underestimates, how often the data were recorded, and whether the model holds.

Grade A* — **Combine calculus with numerical methods**
For example, set a derivative equal to zero to find a maximum, then solve the resulting equation numerically.

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