Numerical integration: the trapezium rule

P9.3 · Pure mathematics · revision notes

The trapezium rule estimates $\int_a^b y\,\mathrm{d}x$ by splitting the area into strips and treating each strip as a trapezium. You need to complete a table of values, apply the formula, decide from the shape of the curve whether the estimate is too big or too small, and say how to improve it. Questions are usually 3 to 6 marks, sometimes followed by a 'hence' part.

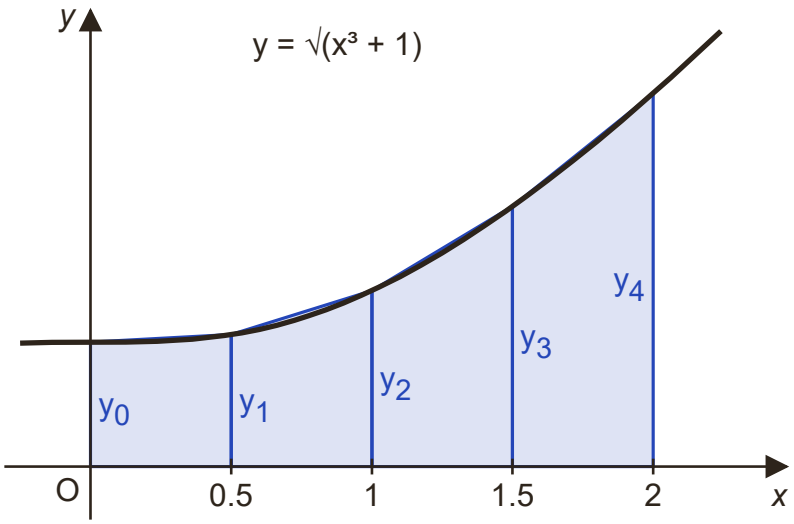
Key facts

- $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0+y_n)+2(y_1+\cdots+y_{n-1})\}$
- $h=\frac{b-a}{n}$; n strips, $n+1$ ordinates
- Concave ($\mathrm{f}''<0$): underestimate
- $\int_a^b(\mathrm{f}(x)+k)\,\mathrm{d}x=\int_a^b \mathrm{f}(x)\,\mathrm{d}x+k(b-a)$
- Percentage error $=\frac{|\text{estimate}-\text{exact}|}{\text{exact}}\times 100\%$
- Convex ($\mathrm{f}''>0$): overestimate
- More strips (smaller h): more accurate

Notes

The rule

- $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0+y_n)+2(y_1+\cdots+y_{n-1})\}$, where $h=\frac{b-a}{n}$. It is in the formula booklet.
- n strips need $n+1$ ordinates (y-values): the first and last are used once, all the others twice.

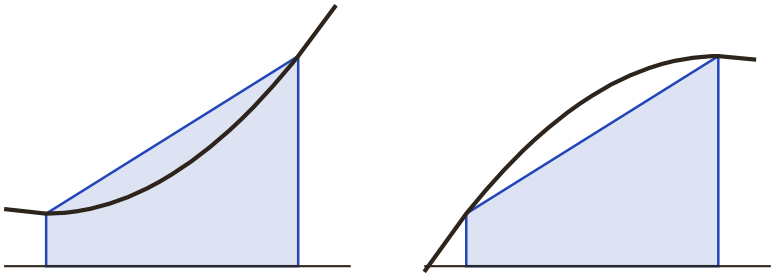


4 strips of width $h=0.5$ need 5 ordinates: y_0 and y_4 are used once, y_1, y_2 and y_3 twice.

- Keep table values to the accuracy given (often 4 or 5 d.p.) and round only the final answer.

Overestimate or underestimate?

- If the curve is **convex** on the interval (bends upwards, $\mathrm{f}''(x)>0$), the top of each trapezium lies above the curve: the rule overestimates.
- If the curve is **concave** (bends downwards, $\mathrm{f}''(x)<0$), the trapezia lie below the curve: the rule underestimates.



convex: overestimate **concave: underestimate**

Only the bending matters: on a convex curve the straight top of each trapezium is above the curve; on a concave curve it is below.

- Whether the curve is increasing or decreasing does not matter, only which way it bends. If it changes concavity in the interval, you cannot tell without more work.

Improving and using the estimate

- More strips (smaller h) give a better estimate.
- Bounds: if one method gives an underestimate and another an overestimate (e.g. lower rectangles and the trapezium rule on a convex, increasing curve), the exact value lies between them.
- 'Hence' parts use properties of integrals: $\int_a^b k\,\mathrm{f}(x)\,\mathrm{d}x = k\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ and $\int_a^b (\mathrm{f}(x)+k)\,\mathrm{d}x = \int_a^b \mathrm{f}(x)\,\mathrm{d}x + k(b-a)$. Log laws such as $\ln x^2 = 2\ln x$ also help.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Complete a table and apply the trapezium rule 5 marks **Grade D**

1. Fill in the missing values with your calculator, to the accuracy asked.
2. Find h from the table (the gap between x-values).
3. Substitute into the formula, showing the bracket, and give the answer to the accuracy asked.

Example. The table shows some values of $y = \sqrt{x^3+1}$ for $0 \le x \le 2$.

x	0	0.5	1	1.5	2
y	1		1.4142		3

(a) Complete the table, giving values to 4 decimal places.

(b) Use the trapezium rule with all the values in the table to estimate $\int_0^2 \sqrt{x^3+1}\,\mathrm{d}x$, giving your answer to 3 significant figures.

Model answer.

(a)

x	0	0.5	1	1.5	2
y	1	**1.0607**	1.4142	**2.0917**	3

B1 B1

(one for each new value)

(b) $h = 0.5$ **B1**

$$\frac{1}{2} \times 0.5 \times [1 + 3 + 2(1.0607 + 1.4142 + 2.0917)]$$ **M1**

$$= 0.25 \times 13.1332$$
$$= 3.28 \quad \text{(3 s.f.)}$$ **A1**

Decide whether the estimate is too big or too small

1 to 2 marks | **Grade B**

1. Decide which way the curve bends on the interval (from a sketch, or the sign of $\mathbf{f''(x)}$).
2. State overestimate (convex) or underestimate (concave).
3. Give the reason: where the tops of the trapezia lie compared with the curve.

Example. The curve $y = \sqrt{x^3 + 1}$ is convex (bends upwards) for $0 \leq x \leq 2$. The trapezium rule gives 3.28 as an estimate of $\int_0^2 \sqrt{x^3 + 1}\,\mathrm{d}x$.

State, with a reason, whether this is an overestimate or an underestimate.

Model answer. Overestimate **B1**, because the curve bends upwards, so the top edge of each trapezium lies above the curve and each trapezium has a larger area than the strip under the curve **B1**

Use the estimate in a 'hence' question

2 marks | **Grade A**

1. Split the new integral into parts you can relate to the one already estimated.
2. Integrate any constant exactly and multiply the estimate by any constant factor.

Example. Given that $\int_0^2 \sqrt{x^3 + 1}\,\mathrm{d}x \approx 3.2833$, estimate $\int_0^2 \left(3 + 2\sqrt{x^3 + 1}\right)\mathrm{d}x$

Model answer.

$\int_0^2 3\,\mathrm{d}x + 2\int_0^2 \sqrt{x^3 + 1}\,\mathrm{d}x \approx 6 + 2 \times 3.2833$ **M1**

$= 12.6$ (3 s.f.) **A1**

Find the percentage error 5 marks **Grade A**

1. Apply the trapezium rule as usual.
2. Integrate exactly to find the true value.
3. Percentage error = difference ÷ exact value × 100, using unrounded values.

Example. (a) Use the trapezium rule with 4 strips to estimate $\int_1^3 \frac{1}{x}\,\mathrm{d}x$, giving your answer to 3 decimal places.
(b) Find the percentage error in your estimate.

Model answer.

(a) $h = 0.5$

x	1	1.5	2	2.5	3
y	1	0.6667	0.5	0.4	0.3333

B1

$$\frac{1}{2} \times 0.5$$
$$\times [1 + 0.3333 + 2(0.6667 + 0.5 + 0.4)]$$ **M1**
$$= 1.117 \text{ (3 d.p.)}$$ **A1**

(b) The exact value is $\ln 3 = 1.0986\ldots$, so the percentage error is

$$\frac{1.1166\ldots - 1.0986\ldots}{1.0986\ldots} \times 100$$ **M1**
$$= 1.64\% \text{ (3 s.f.)}$$ **A1**

Don't lose marks

- Using the number of ordinates instead of the number of strips to find h
- Doubling the end values, or forgetting to double the middle ones.
- Missing the $\frac{1}{2}$, i.e. using h instead of $\frac{1}{2}h$
- Saying 'overestimate because the curve is increasing': the reason must be about which way the curve bends.
- Working in degrees for trigonometric functions in the table.

More tips

Memory tricks

- 'Ends once, middles twice, times half the strip width.'
- A convex curve looks like a smile: the tops of the trapezia sit above it, so the rule overestimates. Concave (a frown): underestimate.
- Count check: 4 strips means 5 values of y
- Store table values in the calculator's memory to avoid retyping errors.

Exam technique

- Write the formula with your numbers substituted before the final value: it earns the method mark.
- Use all the values in the table unless told otherwise, and give the answer to the accuracy asked.
- For 'how could the estimate be improved?', say 'use more strips' (a smaller h); 'use a calculator' earns nothing.
- If a sketch is given, use its shape to judge over- or underestimate; otherwise find $\mathrm{f}''(x)$ or reason from a graph you know.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E

Complete a table of values

Substitute each x into the function (radians for trigonometric functions) and round as instructed.

Grade D

Apply the trapezium rule formula

$\frac{1}{2}h\left[(y_0 + y_n) + 2(y_1 + \cdots + y_{n-1})\right]$ with the values from the table.

Grade C

Find the strip width h

$h = \frac{b-a}{n}$ for n strips; there are $n+1$ values of y

Grade B

State over- or underestimate with a reason

Convex curve (bending upwards): overestimate. Concave curve (bending downwards): underestimate.

Grade B

Explain how to improve the estimate

Use more strips (a smaller h), so the trapezia fit the curve more closely.

Grade A

Use the estimate for a related integral

For example $\int_a^b (k + \mathrm{f}(x))\,\mathrm{d}x = k(b-a) + \int_a^b \mathrm{f}(x)\,\mathrm{d}x$

Grade A

Compare an estimate with the exact value

Percentage error $= \frac{|\text{estimate} - \text{exact}|}{\text{exact}} \times 100\%$

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