The Newton–Raphson method

P9.2b · Pure mathematics · revision notes

The Newton–Raphson method finds roots of $\mathrm{f}(x) = 0$ with the iteration $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$ (in the formula booklet), where each step follows the tangent at x_n down to the x-axis. It usually converges quickly, but it fails when $\mathrm{f}'(x_n)$ is zero or close to zero. Questions are 3 to 5 marks: apply the method, or explain why it fails.

Key facts

- $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$
- Tangent at x_n: $y - \mathrm{f}(x_n) = \mathrm{f}'(x_n)(x - x_n)$
- Fails if $\mathrm{f}'(x_n) = 0$ (horizontal tangent)
- Radians for trigonometric functions
- x_{n+1} is where the tangent at x_n meets the x-axis
- Unreliable near a stationary point ($\mathrm{f}'(x_n)$ close to 0)

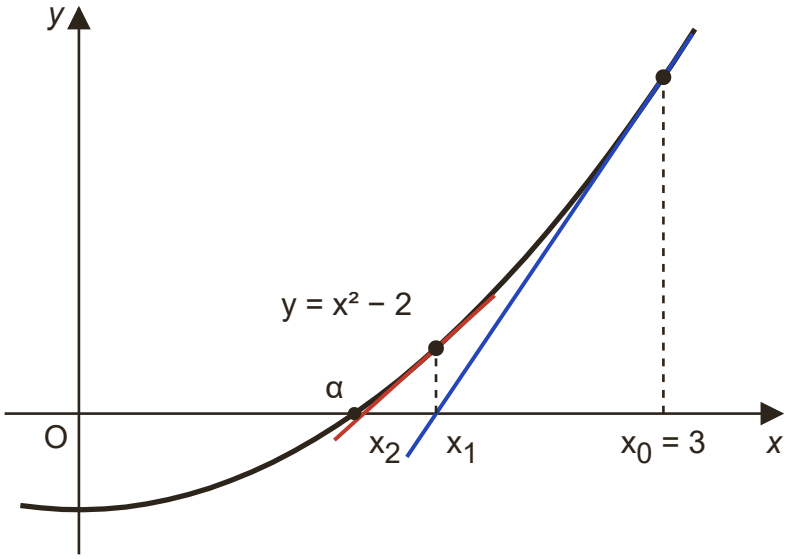
Notes

The formula

- $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$. The equation must be in the form $\mathrm{f}(x) = 0$ first.
- The formula is in the booklet, but you must differentiate f yourself.

Why it works

- The tangent at $(x_n, f(x_n))$ has equation $y - f(x_n) = f'(x_n)(x - x_n)$
- Setting $y = 0$ gives $x = x_n - \dfrac{f(x_n)}{f'(x_n)}$: the next approximation is where the tangent meets the x-axis.



Each new approximation is where the tangent at the previous one meets the x-axis: $3 \to 1.833 \to 1.462 \to$ the root $\sqrt{2} = 1.414$

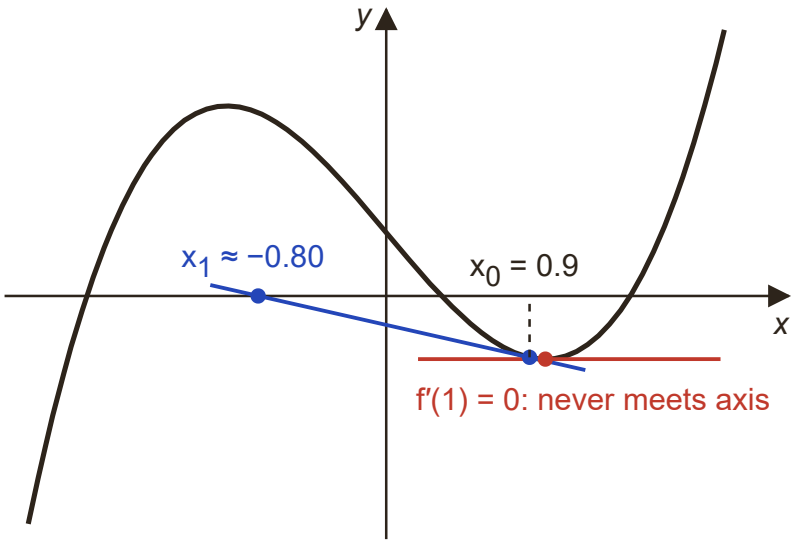
- Close to a root the tangent is a good fit to the curve, so the approximations improve very quickly.

Using it

- Calculator: enter x_0, then type ANS $- \dfrac{f(\text{ANS})}{f'(\text{ANS})}$ and press = repeatedly.
- Trigonometric functions must be in radians: the derivatives of $\sin x$ and $\cos x$ only hold in radians.

When it fails

- If $\mathrm{f}'(x_n) = 0$, the formula divides by zero: the tangent is horizontal and never crosses the x-axis.
- If x_n is near a stationary point, $\mathrm{f}'(x_n)$ is small, so the tangent is shallow and x_{n+1} lands far away. It may then converge to a different root, or leave the domain (e.g. a negative value in $\ln x$).



Near a stationary point the tangent is shallow: from $x_0 = 0.9$ it lands at $x_1 \approx -0.80$, far from the nearest root.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Apply Newton–Raphson once 3 marks **Grade C**

1. Differentiate $\mathrm{f}(x)$
2. Work out $\mathrm{f}(x_0)$ and $\mathrm{f}'(x_0)$ and substitute into the formula, showing the numbers.
3. Give x_1 to the accuracy asked.

Example. $\mathrm{f}(x) = 2x^3 - 5x + 1$. The equation $\mathrm{f}(x) = 0$ has a root near $x = 1.5$. Taking $x_0 = 1.5$ as a first approximation, apply the Newton–Raphson method once to find a second approximation, giving your answer to 3 decimal places.

Model answer.

$\mathrm{f}'(x) = 6x^2 - 5$ **B1**

$\mathrm{f}(1.5) = 0.25$ and $\mathrm{f}'(1.5) = 8.5$, so $x_1 = 1.5 - \dfrac{0.25}{8.5}$ **M1**

$x_1 = 1.471$ (3 d.p.) **A1**

Iterate with a trigonometric function

5 marks | Grade B

1. Set the calculator to radians and differentiate.
2. Apply the formula with x_0, then again with the unrounded x_1
3. Round only the values you write down.

Example. $\mathrm{f}(x) = 2\sin x - x$, where x is in radians. The equation $\mathrm{f}(x) = 0$ has a root near $x = 2$.
Use the Newton–Raphson method with $x_0 = 2$ to find x_1 and x_2, giving your answers to 4 decimal places.

Model answer.

$\mathrm{f}'(x) = 2\cos x - 1$ **B1**

$x_1 = 2 - \dfrac{2\sin 2 - 2}{2\cos 2 - 1}$

$= 2 - \dfrac{-0.181\ldots}{-1.832\ldots}$ **M1**

$x_1 = 1.9010$ **A1**

$x_2 = x_1 - \dfrac{\mathrm{f}(x_1)}{\mathrm{f}'(x_1)}$, using the unrounded x_1 **M1**

$x_2 = 1.8955$ **A1**

Explain why a starting value is unsuitable

2 marks | Grade A

1. Work out $\mathrm{f}'(x_0)$
2. If it is zero (or very small), explain in terms of the tangent: horizontal (or nearly), so it does not meet the x-axis (or meets it far away).

Example. $\mathrm{g}(x) = x^3 - 3x + 1$.
Explain why $x_0 = 1$ cannot be used as a starting value for the Newton–Raphson method to solve $\mathrm{g}(x) = 0$

Model answer.

$\mathrm{g}'(x) = 3x^2 - 3$, so $\mathrm{g}'(1) = 0$ **M1**

The formula would divide by zero: $x = 1$ is a stationary point, so the tangent there is horizontal and never meets the x-axis **A1**

Don't lose marks

- Differentiating wrongly, especially trigonometric, exponential or log terms.
- Using degrees mode.

- Rounding x_1 before working out x_2
- Dropping brackets when $f(x_n)$ or $f'(x_n)$ is negative, which flips a sign.
- Writing only 'the method fails' without the reason: $f'(x_0) = 0$, so the tangent is parallel to the x-axis.

More tips

Memory tricks

- 'Tangent to the axis': follow the tangent at x_n down to the x-axis to get x_{n+1}
- Use ANS in the calculator so each step uses the full, unrounded value.
- Once it is close to a root, the method roughly doubles the number of correct decimal places each step, so two or three steps are usually enough.
- Check your derivative before iterating: one wrong sign spoils every value.

Exam technique

- Show the substitution with numbers (e.g. $1.5 - \frac{0.25}{8.5}$) before the answer: it earns the method mark even if you press a wrong key.
- Write $f'(x)$ on its own line; it is often a mark by itself.
- When asked to explain a failure, refer to the tangent and the stationary point, not just 'division by zero'.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade D — **Differentiate f(x) accurately**
A correct $f'(x)$ is the first mark; use the chain, product or quotient rule where needed.

Grade C — **Apply one Newton–Raphson step**
Substitute x_0 into $x_0 - \frac{f(x_0)}{f'(x_0)}$, showing the values of $f(x_0)$ and $f'(x_0)$

Grade B — **Iterate to the accuracy required**
Repeat using ANS until the values agree to the accuracy asked.

Grade B — **Explain the method using tangents**
x_{n+1} is where the tangent to the curve at $x = x_n$ crosses the x-axis.

Grade A — **Explain why a starting value fails**
If $f'(x_0) = 0$ the tangent is horizontal and never meets the axis; near a stationary point, x_1 can land far away.

Grade A* — **Derive the formula from the tangent**
Write the tangent $y - f(x_n) = f'(x_n)(x - x_n)$, put $y = 0$ and solve for x

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