Iteration and staircase/cobweb diagrams

P9.2a · Pure mathematics · revision notes

To solve $f(x) = 0$ numerically, you can rearrange it into the form $x = g(x)$ and use the iteration $x_{n+1} = g(x_n)$. The values may converge to a root, shown by a staircase or cobweb diagram, or they may diverge. Questions typically ask you to show a rearrangement, calculate iterates, then confirm a root or explain what the iteration does.

Key facts

- Iteration: $x_{n+1} = g(x_n)$
- Roots of $f(x) = 0$ are where $y = x$ meets $y = g(x)$
- Diagram: vertically to the curve, horizontally to $y = x$
- Staircase: approaches the root from one side
- Cobweb: alternates either side of the root
- Grade A or above only Converges near α if $|g'(\alpha)| < 1$; moves away if $|g'(\alpha)| > 1$
- Keep full accuracy with the ANS key

Notes

Rearranging

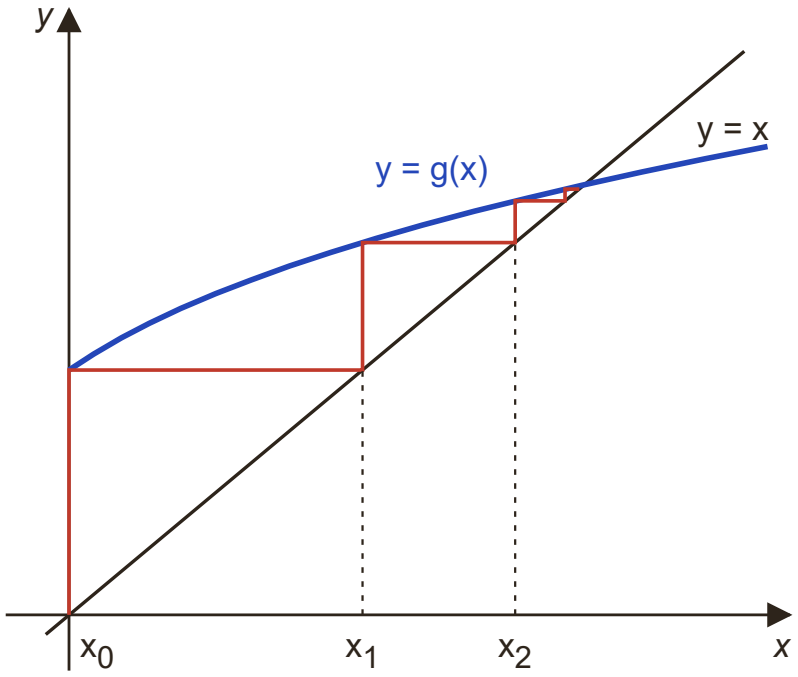
- For any rearrangement $x = g(x)$ of $f(x) = 0$, the roots are where the graphs of $y = x$ and $y = g(x)$ meet.
- There are many possible rearrangements. $x^3 - 4x - 2 = 0$ can become $x = \sqrt{4 + \frac{2}{x}}$, $x = \frac{x^3 - 2}{4}$ or $x = \sqrt[3]{4x + 2}$. Some converge to a particular root and some do not.

Iterating

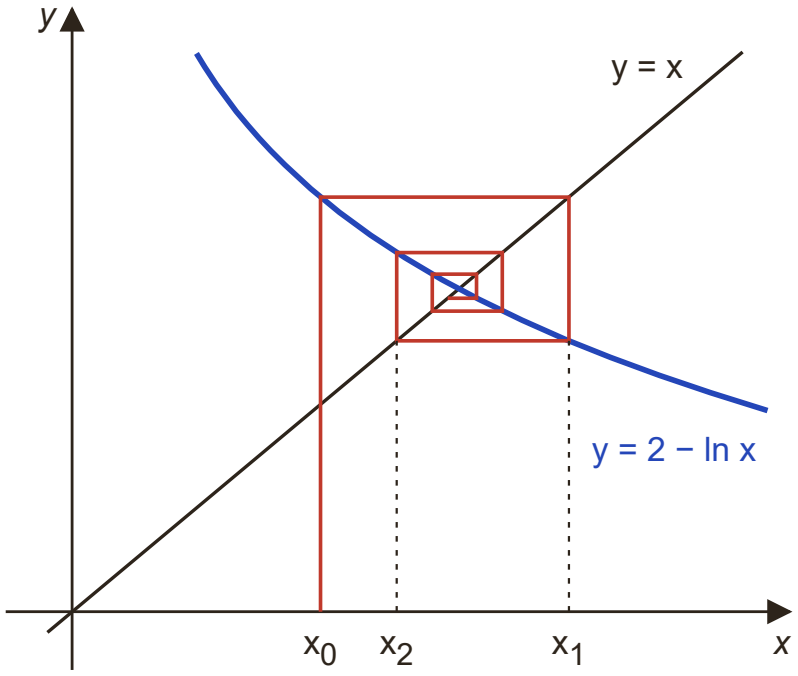
- Start with x_0, then $x_1 = g(x_0)$, $x_2 = g(x_1)$, and so on.
- On a calculator: type x_0 and press =, type the formula with ANS in place of x, then press = repeatedly. This keeps full accuracy; round only the values you write down.
- If the values settle down to a limit, that limit is a root of the equation.

Staircase and cobweb diagrams

- Draw $y = g(x)$ and $y = x$. From x_0 on the x-axis go vertically to the curve, then horizontally to the line $y = x$; you are now directly above x_1. Repeat.
- Staircase: the values approach the root from one side. Cobweb: they alternate from one side of the root to the other, spiralling in.



Staircase: $x_{n+1} = \sqrt[3]{4x_n + 2}$ from $x_0 = 0$ climbs towards the root from one side ($0 < g'(\alpha) < 1$).



Cobweb: $x_{n+1} = 2 - \ln x_n$ from $x_0 = 1$ jumps from side to side of the root, closing in ($-1 < g'(\alpha) < 0$).

- Near the root α: if $0 < g'(\alpha) < 1$ you get a converging staircase; if $-1 < g'(\alpha) < 0$, a converging cobweb; if $|g'(\alpha)| > 1$, the values move away from α Grade A or above only

How iteration fails

- The values can diverge, or converge to a different root from the one you want.
- A starting value far from the root can lead to another root, or to a value outside the domain of g (e.g. the square root of a negative number).

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, A1* = the solution must be fully correct, with every step shown.

Show that an equation can be rearranged 2 marks Grade D

1. Start from the given equation and aim for the form in the question.
2. Show every step, including any division or square root.
3. State any condition used, such as $x > 0$ when taking the positive square root.

Example. Show that the equation $x^3 - 4x - 2 = 0$ can be written in the form $x = \sqrt{4 + \frac{2}{x}}$, for $x > 0$

Model answer.

$x^3 = 4x + 2 \Rightarrow x^2 = 4 + \frac{2}{x}$ (dividing by x) **M1**

$x = \sqrt{4 + \frac{2}{x}}$, taking the positive root as $x > 0$ **A1***

Use an iteration formula, then confirm the root

5 marks | Grade C

1. Substitute x_0 and use ANS for each following iterate.
2. Write each value to the accuracy asked.
3. To show a root to a given accuracy, use a change of sign on $\mathrm{f}(x)$ at the rounding bounds.

Example. Use the iteration formula $x_{n+1} = \sqrt{4 + \dfrac{2}{x_n}}$ with $x_0 = 2$ to find x_1, x_2 and x_3, giving each to 4 decimal places.
Hence show that the positive root of $x^3 - 4x - 2 = 0$ is 2.214 correct to 3 decimal places.

Model answer.

$x_1 = \sqrt{4+1} = 2.2361$ **M1 A1**

$x_2 = 2.2123$, $x_3 = 2.2145$ **A1**

Let $\mathrm{f}(x) = x^3 - 4x - 2$: $\mathrm{f}(2.2135) = -0.0088\ldots < 0$ and $\mathrm{f}(2.2145) = 0.0019\ldots > 0$ **M1**

Change of sign and f continuous, so the root is 2.214 (3 d.p.) **A1**

Explain why an iteration does not find the root

3 marks | Grade B

1. Work out the first few iterates.
2. Describe what they do: moving away from the root, or heading to a different root.
3. Give the reason in terms of the graph: near the root, $y = \mathrm{g}(x)$ is steeper than $y = x$

Example. A student uses $x_{n+1} = \dfrac{x_n^3 - 2}{4}$ with $x_0 = 2$, hoping to find the root of $x^3 - 4x - 2 = 0$ near 2.214.
Find x_1 and x_2, and explain why this iteration does not find that root.

Model answer.

$x_1 = \dfrac{8-2}{4} = 1.5$ and $x_2 = \dfrac{3.375 - 2}{4} = 0.34375$ **M1 A1**

The values move away from 2.214 (they go on to converge to a different, negative root), because near 2.214 the curve $y = \dfrac{x^3 - 2}{4}$ is steeper than the line $y = x$: its gradient there is about 3.7 **B1**

Predict a staircase or a cobweb

2 marks **Grade A**

1. Differentiate $g(x)$ and evaluate the gradient near the root.
2. Between 0 and 1: converging staircase. Between -1 and 0: converging cobweb. Size greater than 1: moves away.

Example. The iteration $x_{n+1} = 2 - \ln x_n$ converges to the root $\alpha \approx 1.557$ of $x + \ln x = 2$. Explain whether the iteration gives a staircase or a cobweb diagram.

Model answer.

$g(x) = 2 - \ln x$, so $g'(x) = -\dfrac{1}{x}$ and $g'(1.557) \approx -0.64$ **M1**

This is between -1 and 0, so the values converge while alternating either side of α: a cobweb diagram **A1**

Don't lose marks

- Rounding each iterate before using it in the next step.
- Giving iterates to the wrong number of decimal places.
- Drawing the diagram from the line first instead of the curve, or joining points with sloping lines.
- Claiming a root is correct to 3 d.p. from iterates alone, without the change of sign when it is asked for.
- Forgetting that a rearrangement may converge to a different root from the one wanted.

More tips

Memory tricks

- The ANS key keeps every digit, so your iterates match the mark scheme.
- Staircase or cobweb? Look at the sign of the gradient of $y = g(x)$ near the root: positive gives a staircase, negative a cobweb.
- Sense check: substitute your final value into $f(x)$; the result should be close to 0.
- In diagrams, always 'up to the curve, across to the line'.

Exam technique

- For 'show that' rearrangements, write every step and state any condition used.
- Write each iterate on its own, labelled x_1, x_2, …, to the accuracy asked.
- On a diagram, mark x_0, x_1, x_2 on the x-axis and draw arrows to show the direction.
- An iteration alone never proves accuracy: to show a root to a given number of decimal places, use a change of sign.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E **Calculate iterates from a formula**

Substitute x_0 to get x_1, then x_1 to get x_2, and so on, using the ANS key.

Grade D

Rearrange f(x) = 0 into x = g(x)

Use algebra to isolate an x on one side, e.g. $x^3 - 4x - 2 = 0 \Rightarrow x = \sqrt{4 + \frac{2}{x}}$

Grade C

Draw staircase and cobweb diagrams

Go vertically to the curve $y = g(x)$, then horizontally to the line $y = x$, and repeat.

Grade C

Confirm a root with a sign change

After iterating, use a change of sign in $f(x)$ to show the root to the accuracy required.

Grade B

Explain convergence or divergence from iterates

Say whether the values approach a limit, and whether it is the root you want.

Grade A

Use the gradient of g to predict behaviour

Near the root, $|g'(x)| < 1$ gives convergence: a staircase if g' is positive, a cobweb if it is negative.

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