Locating roots by change of sign

P9.1 · Pure mathematics · revision notes

If a function is continuous on an interval and changes sign, the equation $\mathbf{f}(x) = 0$ has a root in that interval. You use this to show that a root lies between two values, or that an approximation is correct to a given accuracy, and you must know when the method fails. Questions are usually 2 or 3 marks, often at the start of a longer numerical methods question.

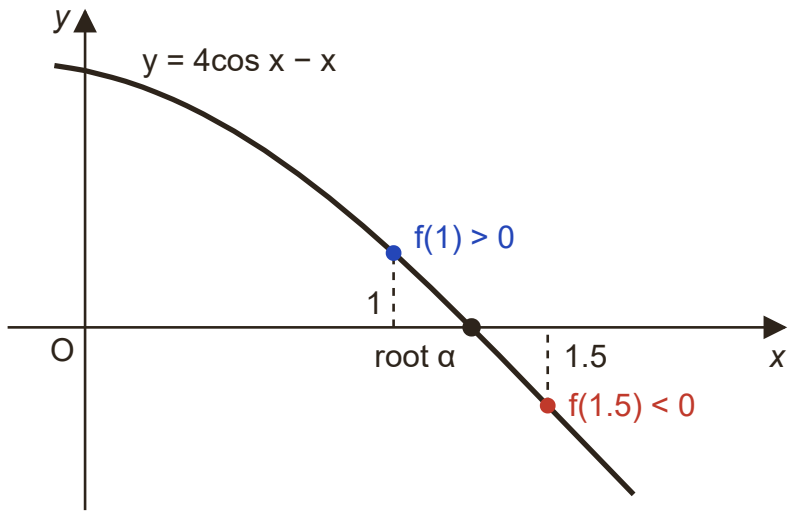
Key facts

- $\mathbf{f}$ continuous on $[a, b]$ and $\mathbf{f}(a)$, $\mathbf{f}(b)$ of opposite signs $\Rightarrow$ a root between a and b
- Root $= 2.37$ (2 d.p.): test $x = 2.365$ and $x = 2.375$
- No sign change, but roots: an even number of roots, or a repeated root (curve touches the axis)
- To solve $\mathbf{g}(x) = \mathbf{h}(x)$, use $\mathbf{f}(x) = \mathbf{g}(x) - \mathbf{h}(x)$
- Sign change, no root: discontinuity (asymptote)
- Trigonometric functions: calculator in radians

Notes

The change of sign rule

- If $\mathbf{f}$ is continuous on $[a, b]$, and $\mathbf{f}(a)$ and $\mathbf{f}(b)$ have opposite signs, there is at least one root of $\mathbf{f}(x) = 0$ with $a < x < b$
- Continuous means the graph has no breaks or jumps on the interval, so it must cross the axis to get from below it to above it.



f is continuous and changes sign between 1 and 1.5, so the curve must cross the x-axis in between.

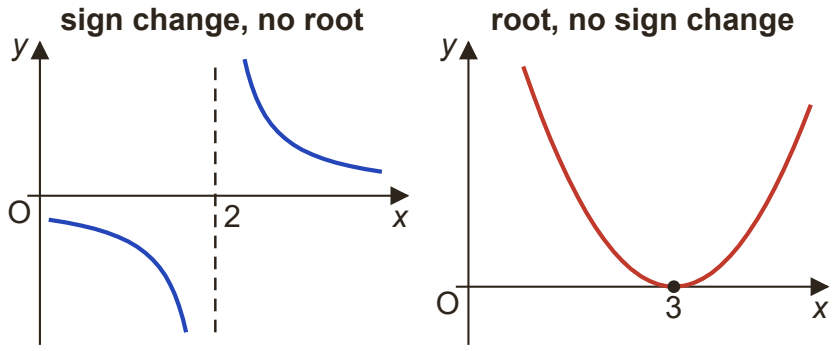
- A full answer has three parts: the two values (showing their signs), 'there is a change of sign and $\mathbf{f}$ is continuous', and 'so there is a root in the interval'.
- For an equation such as $\mathbf{g}(x) = \mathbf{h}(x)$, first write it as $\mathbf{f}(x) = \mathbf{g}(x) - \mathbf{h}(x) = 0$

Showing a root to a given accuracy

- To show that a root $\alpha = 1.252$ to 3 d.p., test the ends of the interval of numbers that round to 1.252: $\mathbf{f}(1.2515)$ and $\mathbf{f}(1.2525)$
- A change of sign means $1.2515 < \alpha < 1.2525$, so $\alpha = 1.252$ to 3 d.p.

When the method fails

- Sign change but no root: the function is not continuous, e.g. $\mathbf{f}(x) = \dfrac{1}{x-2}$ changes sign across the asymptote $x = 2$
- Root but no sign change: two roots (or any even number) in the interval, or a repeated root where the curve touches the axis, e.g. $\mathbf{f}(x) = (x-3)^2$



Left: $\dfrac{1}{x-2}$ jumps across the asymptote $x = 2$. Right: $(x-3)^2$ touches the axis at a repeated root and never goes negative.

- A small interval reduces these risks, and a sketch shows how many roots to expect.

Calculator

- Use radians for trigonometric functions. Table mode finds sign changes quickly. Give values to at least 2 significant figures so the signs are clear.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Show that a root lies in an interval — 2 marks — Grade E

1. Write the equation as $\mathbf{f}(x) = 0$ if it is not already.
2. Work out $\mathbf{f}$ at both ends of the interval and write the values down.
3. Conclude: change of sign, $\mathbf{f}$ continuous, so a root in the interval.

Example. $\mathbf{f}(x) = 4\cos x - x$, where x is in radians.
Show that the equation $\mathbf{f}(x) = 0$ has a root α in the interval $[1, 1.5]$

Model answer.

$\mathbf{f}(1) = 1.161\ldots$ and $\mathbf{f}(1.5) = -1.217\ldots$ **M1**

There is a change of sign and $\mathbf{f}$ is continuous, so there is a root α in $[1, 1.5]$ **A1**

Show that a root is correct to a given accuracy

3 marks | Grade C

1. Find the interval of numbers that round to the given value (half a unit either side).
2. Work out f at both bounds, keeping enough figures to show the signs.
3. Conclude with the change of sign, continuity and the rounded root.

Example. $\mathrm{f}(x) = 4\cos x - x$, where x is in radians. The equation $\mathrm{f}(x) = 0$ has a root α. Show that $\alpha = 1.252$ correct to 3 decimal places.

Model answer.

Use the interval $[1.2515, 1.2525]$ **M1**

$\mathrm{f}(1.2515) = 0.0041\ldots > 0$ and $\mathrm{f}(1.2525) = -0.0007\ldots < 0$ **A1**

There is a change of sign and f is continuous, so $1.2515 < \alpha < 1.2525$ and $\alpha = 1.252$ (3 d.p.) **A1**

Explain why a sign change does not give a root

2 marks | Grade B

1. Look for a value in the interval where the function is undefined (a vertical asymptote).
2. Say that the function is not continuous on the interval, so the rule does not apply.
3. If you can, find the actual root(s) to show none is in the interval.

Example. $\mathrm{g}(x) = \dfrac{3}{2x-3} + 1$. A student notes that $\mathrm{g}(1) = -2$ and $\mathrm{g}(2) = 4$, and concludes that $\mathrm{g}(x) = 0$ has a root between 1 and 2.
Explain why the student is wrong.

Model answer.

g is not continuous on $[1, 2]$: its graph has a vertical asymptote at $x = 1.5$ **B1**

Solving $\dfrac{3}{2x-3} = -1$ gives $x = 0$, so the only root is outside the interval; the sign change is caused by the asymptote **B1**

Don't lose marks

- Leaving out 'continuous' or the final conclusion, which loses the accuracy mark.
- Testing the wrong interval, e.g. 1.25 and 1.26 when showing that a root is 1.252 to 3 d.p.
- Calculator in degrees for trigonometric functions.
- Rounding a small value such as -0.0007 to 0, so the sign cannot be seen.
- Testing signs of $\mathrm{g}(x)$ and $\mathrm{h}(x)$ separately instead of $\mathrm{f}(x) = \mathrm{g}(x) - \mathrm{h}(x)$

More tips

Memory tricks

- The three-part conclusion: values with signs; 'change of sign, continuous'; 'so a root'.
- Bounds for rounding are half a unit either side of the last decimal place: 1.252 gives 1.2515 and 1.2525.
- Use the calculator's table mode to scan for a sign change.
- Failure cases: an asymptote gives a false sign change; a touching curve or two close roots hide a root.

Exam technique

- Write down the actual values (at least 2 s.f.), not just 'positive' and 'negative'.
- Use the notation of the question: if the root is called α, say 'α lies in …'.
- For 'explain why' questions, name the feature (asymptote, repeated root, two roots) and link it to the graph.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Show a root lies in an interval**
Work out $\mathrm{f}(a)$ and $\mathrm{f}(b)$, show they have opposite signs, and conclude using continuity.

Grade D — **Rearrange an equation into f(x) = 0**
To find where $\mathrm{e}^{x} = 4 - x$, use $\mathrm{f}(x) = \mathrm{e}^{x} + x - 4$

Grade C — **Show a root is correct to given accuracy**
Test the bounds of the rounding interval, e.g. 1.2515 and 1.2525 for a root of 1.252 to 3 d.p.

Grade B — **Explain how a sign change can mislead**
A vertical asymptote (a break in the graph) can give a sign change with no root.

Grade B — **Explain how roots can be missed**
With no sign change there may still be two roots, or a root where the curve just touches the axis.

Grade A — **Count roots using a sketch**
Sketch $y = \mathrm{g}(x)$ and $y = \mathrm{h}(x)$: the number of intersections is the number of roots of $\mathrm{g}(x) = \mathrm{h}(x)$

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