Integration as the limit of a sum

P8.4 · Pure mathematics · revision notes

A definite integral is the limit of a sum of the areas of thin rectangles: $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \lim_{\delta x \to 0} \sum_{x=a}^{b} \mathrm{f}(x)\,\delta x$. Questions are short: write a limit of a sum as an integral and evaluate it, use rectangles to find bounds for an area, or explain what the sum means in a context.

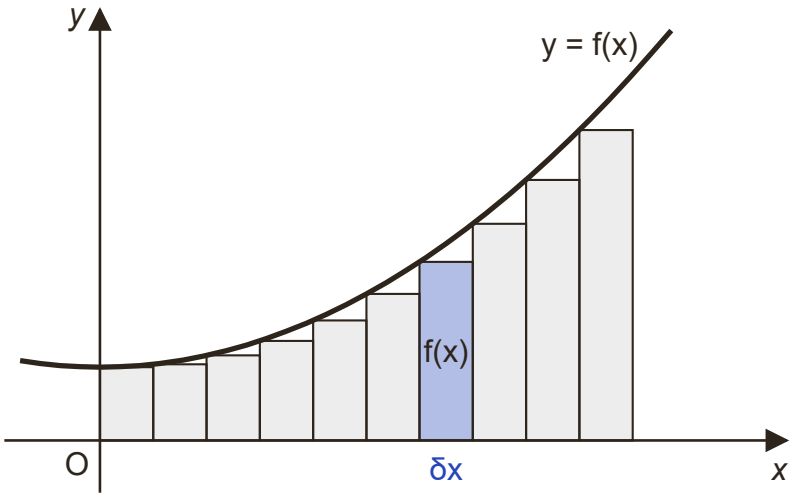
Key facts

- $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \lim_{\delta x \to 0} \sum_{x=a}^{b} \mathrm{f}(x)\,\delta x$
- One strip: area $\approx \mathrm{f}(x)\,\delta x$ (height × width)
- $\sum \to \int$ and $\delta x \to \mathrm{d}x$; same function, same limits
- Increasing function: left heights give the lower sum, right heights the upper sum
- Decreasing function: left heights give the upper sum, right heights the lower sum
- Lower sum $\leq \int_a^b \mathrm{f}(x)\,\mathrm{d}x \leq$ upper sum
- Width of each of n strips: $h = \dfrac{b-a}{n}$

Notes

The idea

- Split the interval $a \leq x \leq b$ into thin strips of width δx. Each strip is roughly a rectangle of height $\mathrm{f}(x)$, so its area is about $\mathrm{f}(x)\,\delta x$



Each strip is roughly a rectangle of area $f(x)\,\delta x$; adding them all estimates the area.

- Adding the strips estimates the area under the curve: $\sum_{x=a}^{b} \mathrm{f}(x)\,\delta x$
- As $\delta x \to 0$ the rectangles fit the curve better and better, and the sum tends to the exact area, which is the definite integral.

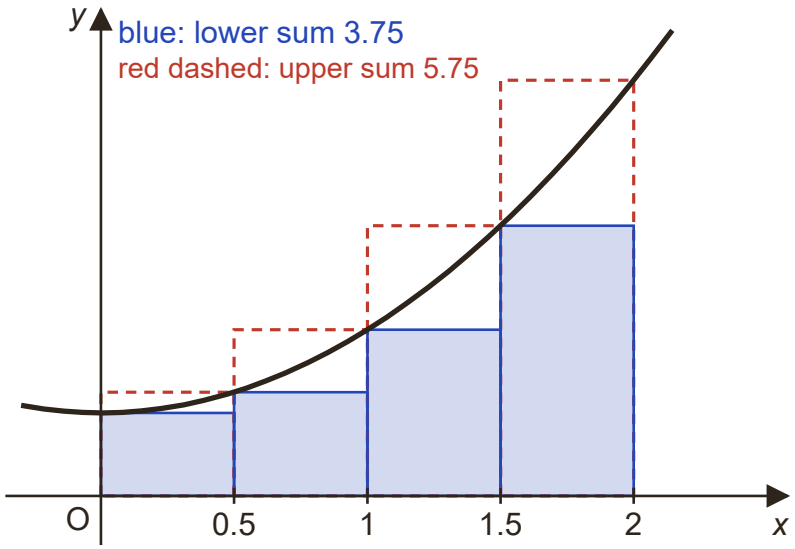
The notation

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \lim_{\delta x \to 0} \sum_{x=a}^{b} \mathrm{f}(x)\,\delta x$$

- The integral sign is a stretched S (for 'sum'), and $\mathrm{d}x$ comes from δx. The function and the limits stay the same.
- The sum need not be an area. In a context, $\mathrm{f}(x)\,\delta x$ might be a small distance ($v\,\delta t$) or a small volume (rate $\times\,\delta t$), and the integral adds them all up.

Upper and lower sums

- On each strip, a lower rectangle uses the least value of f on that strip and an upper rectangle uses the greatest.
- For an increasing function the lower rectangles use the left-hand heights and the upper rectangles use the right-hand heights; for a decreasing function it is the other way round.



For an increasing curve the lower rectangles use left-hand heights and the upper ones right-hand heights: $3.75 < \dfrac{14}{3} < 5.75$

- Lower sum $<$ true area $<$ upper sum. Narrower strips squeeze the two bounds together.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, A1* = the solution must be fully correct, with every step shown.

Write a limit of a sum as an integral and evaluate it

3 to 4 marks **Grade B**

1. Copy the function and the limits into an integral, replacing δx by $\mathrm{d}x$
2. Rewrite the function as powers if needed and integrate.
3. Substitute the limits and give the exact value.

Example. Show that $\lim_{\delta x \to 0} \sum_{x=1}^{9} \sqrt{x}\,\delta x = \frac{52}{3}$

Model answer.

$$\int_1^9 \sqrt{x}\,\mathrm{d}x$$ **B1**

$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^9$$ **M1**

$$= \frac{2}{3}(27-1)$$

$$= \frac{52}{3}$$ **A1***

Find upper and lower bounds using rectangles

4 to 5 marks | **Grade A**

1. Work out the strip width and the value of y at each strip boundary.
2. Decide from the shape (increasing or decreasing) which heights give the lower and upper rectangles.
3. Add the areas of each set of rectangles.
4. State that the true area lies between the two totals.

Example. The region R is bounded by the curve $y = x^2 + 1$, the x-axis and the lines $x = 0$ and $x = 2$. The interval $0 \leq x \leq 2$ is split into 4 strips of width 0.5.

(a) Find a lower bound and an upper bound for the area of R, using rectangles on these strips.

(b) Explain how the bounds could be improved.

Model answer.

(a)

x	0	0.5	1	1.5	2
y	1	1.25	2	3.25	5

B1

y is increasing, so the lower rectangles use the left-hand heights:

$0.5(1 + 1.25 + 2 + 3.25) = 3.75$ **M1 A1**

The upper rectangles use the right-hand heights: $0.5(1.25 + 2 + 3.25 + 5) = 5.75$ **A1**

(b) Use more, narrower strips: as the width tends to zero both sums tend to the exact area,

$\int_0^2 (x^2 + 1)\,\mathrm{d}x = \frac{14}{3}$ **B1**

Interpret a limit of a sum in context

3 marks **Grade B**

1. Say what one small term represents, e.g. rate × a short time δt
2. Explain that adding the terms and letting $\delta t \to 0$ gives the exact total, which is the integral.
3. Evaluate the integral and give units.

Example. Water flows into a tank at a rate of $r = 3 + 0.2t$ litres per minute, where t is the time in minutes.

(a) Explain why $\lim_{\delta t \to 0} \sum_{t=0}^{10} r\,\delta t$ is the volume of water that flows in during the first 10 minutes.

(b) Find this volume.

Model answer.

(a) In a short time δt the volume flowing in is approximately $r\,\delta t$; adding these for $0 \le t \le 10$ and letting $\delta t \to 0$ gives the exact total volume **B1**

(b) $\int_0^{10} (3 + 0.2t)\,\mathrm{d}t = \left[3t + 0.1t^2\right]_0^{10}$ **M1**

$= 30 + 10 = 40$ litres **A1**

Don't lose marks

- Changing the function or the limits when writing the integral.
- Leaving δx inside the integral instead of $\mathrm{d}x$
- Mixing up the upper and lower sums for a decreasing function.
- Using the number of strips as the width: 4 strips on $0 \le x \le 2$ have width 0.5, not 4.
- Giving a decimal from the calculator when the question asks you to show an exact value.

More tips

Memory tricks

- The $\int$ sign is a long S for 'sum', and δx (a small change) becomes $\mathrm{d}x$
- Converting a limit into an integral: change only the symbols; the function and the numbers at the bottom and top stay put.
- Climbing curve: left is low. Falling curve: left is high.
- Check: your lower bound, the exact value and your upper bound should be in increasing order.

Exam technique

- Write the integral explicitly as your first line: it is often worth a mark on its own.
- When you say which rectangles give a bound, give the reason: the function is increasing (or decreasing) on the interval.
- 'Explain' questions need the idea of the strip width tending to zero, not just 'use more rectangles'.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade C — **Interpret f(x)δx as a rectangle's area**
Each term is a thin strip of height $\mathrm{f}(x)$ and width δx

Grade C — **Write a limit of a sum as an integral**
Keep the same function and limits, and replace $\sum$ and δx by $\int$ and $\mathrm{d}x$

Grade B — **Evaluate the limit by integrating**
Integrate and substitute the limits to give an exact value.

Grade A — **Find upper and lower bounds with rectangles**
Use the greatest and least heights on each strip; the true area lies between the two totals.

Grade A — **Explain why narrower strips give better bounds**
As $\delta x \to 0$ the upper and lower sums both approach the value of the integral.

Free notes, Full packs, practice papers and mock exams at chhetriacademy.com. Written for this site; not taken from past papers. Stuck? 1-to-1 tutoring with a free 30-minute trial lesson.