Differential equations in context: interpreting solutions

P8.8 · Pure mathematics · revision notes

Differential equations model how quantities change: growth and decay, cooling, liquids filling or draining, and motion ($v = \frac{\mathrm{d}x}{\mathrm{d}t}$, $a = \frac{\mathrm{d}v}{\mathrm{d}t}$). You solve them, use data to find constants, then interpret the solution: the meaning of terms, long-term behaviour and the model's limitations. These are long questions, often 8 to 12 marks in total.

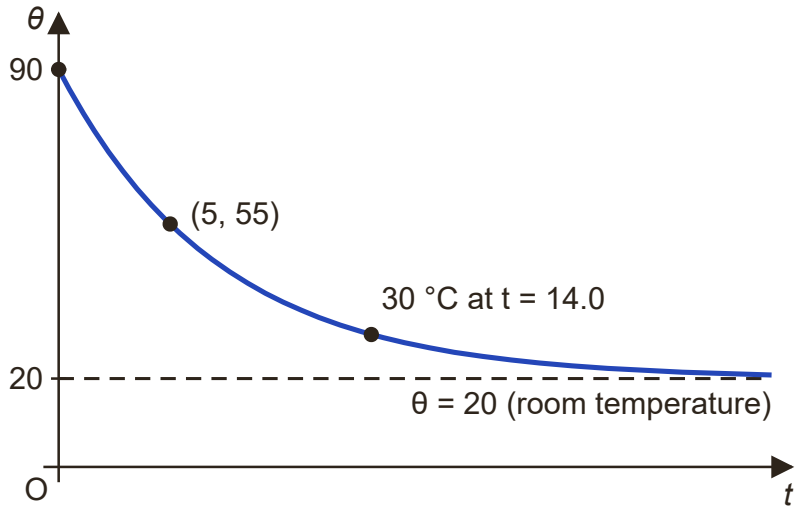
Key facts

- $\frac{\mathrm{d}N}{\mathrm{d}t} = kN \Rightarrow N = N_0\mathrm{e}^{kt}$
- $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - \theta_s) \Rightarrow \theta = \theta_s + (\theta_0 - \theta_s)\mathrm{e}^{-kt}$
- 'Proportional to' means multiply by a constant: rate $= kN$
- Limiting value without solving: set the rate of change equal to 0
- $\mathrm{e}^{-kt} = p \Rightarrow t = -\frac{\ln p}{k}$
- Long term: $\mathrm{e}^{-kt} \to 0$ as $t \to \infty$ (for $k > 0$)
- $v = \frac{\mathrm{d}x}{\mathrm{d}t}$, $a = \frac{\mathrm{d}v}{\mathrm{d}t}$

Notes

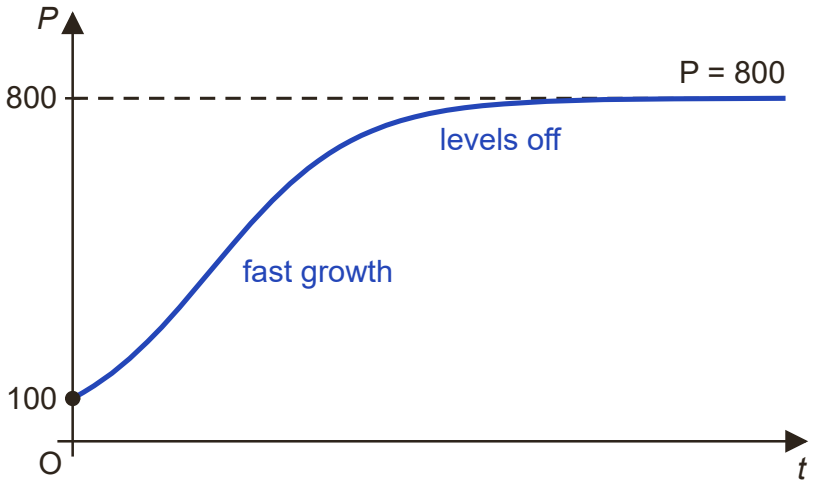
Common models

- Growth or decay at a rate proportional to the amount: $\frac{\mathrm{d}N}{\mathrm{d}t} = kN$ gives $N = N_0\mathrm{e}^{kt}$, where N_0 is the value when $t = 0$
- Cooling (or heating) towards the temperature of the surroundings, θ_s: $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - \theta_s)$ gives $\theta = \theta_s + (\theta_0 - \theta_s)\mathrm{e}^{-kt}$



$\theta = 20 + 70\mathrm{e}^{-kt}$ with $k = \frac{1}{5}\ln 2$: the tea cools quickly at first, then more slowly towards 20 °C.

- Filling and leaking: $\frac{\mathrm{d}V}{\mathrm{d}t}$ = (rate in) − (rate out).
- Limited growth: $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(M - P)$ grows almost exponentially at first, then levels off towards M



Limited growth: almost exponential at first, then levelling off towards the limit (here $P = \dfrac{800}{1 + 7\mathrm{e}^{-0.5t}}$ tends to 800).

- Kinematics: $\dfrac{\mathrm{d}v}{\mathrm{d}t} = -0.5v$ models a particle slowing down; separate the variables to find v, and integrate v to find displacement.

Finding the constants

- Use the initial condition to find the constant of integration, then a second condition to find k
- Keep k exact (e.g. $k = \dfrac{1}{5}\ln 2$) until the final answer to avoid rounding errors.

Interpreting the solution

- The sign of the rate of change tells you whether the quantity is increasing or decreasing.
- Long-term value: let $t \to \infty$ in the solution, or set the rate of change equal to zero in the differential equation.
- Answer in context, with units, e.g. 'the tea cools towards room temperature, 20 °C'.

Evaluating the model

- Compare a prediction with a real value and say whether the model overestimates or underestimates.
- Typical limitations: unlimited growth is impossible (food, space); constants may change (seasons, weather); a continuous model is used for whole numbers of animals; predictions outside the range of the data are unreliable.
- A refinement should fix the specific problem, e.g. add a term that limits growth.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, A1* = the solution must be fully correct, with every step shown.

Solve a model and use it to find a time

8 marks **Grade B**

1. Separate the variables, integrate and use the initial condition.
2. Substitute the second data point and take logs to find k exactly.
3. Substitute the target value, solve for t with logs, and give units.

Example. A cup of tea at 90 °C is left in a room kept at 20 °C. Its temperature, θ °C, t minutes later satisfies $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - 20)$, where k is a positive constant.

(a) Show that $\theta = 20 + 70\mathrm{e}^{-kt}$.

(b) After 5 minutes the temperature is 55 °C. Find the exact value of k.

(c) Find the time taken for the tea to cool to 30 °C.

Model answer.

(a) $\displaystyle\int \frac{1}{\theta - 20}\,\mathrm{d}\theta = \int -k\,\mathrm{d}t$ **M1**

$\ln(\theta - 20) = -kt + c$ **A1**

$t = 0,\ \theta = 90:\ c = \ln 70$ **M1**

$\theta - 20 = \mathrm{e}^{-kt + \ln 70}$

$= 70\mathrm{e}^{-kt}$ so $\theta = 20 + 70\mathrm{e}^{-kt}$ **A1***

(b) $55 = 20 + 70\mathrm{e}^{-5k} \Rightarrow \mathrm{e}^{-5k} = \dfrac{1}{2}$ **M1**, so $k = \dfrac{1}{5}\ln 2$ **A1**

(c) $30 = 20 + 70\mathrm{e}^{-kt} \Rightarrow \mathrm{e}^{-kt} = \dfrac{1}{7} \Rightarrow t = \dfrac{\ln 7}{k} = \dfrac{5\ln 7}{\ln 2}$ **M1**

$t = 14.0$ minutes (3 s.f.) **A1**

Interpret a term and find the long-term value 3 marks **Grade B**

1. Link each term to the story: what is flowing in or out, what is growing or shrinking.
2. For the long term, set the rate of change to zero (or let $t \to \infty$ in a solution).
3. State the value with units and a reason.

Example. Water flows into a tank at a constant rate and leaks out at a rate proportional to the volume in the tank. The volume, V litres, after t minutes satisfies $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 20 - 0.1V$.

(a) Explain what the term $0.1V$ represents.

(b) Without solving the equation, find the volume that the model predicts in the long term. Give a reason.

Model answer.

(a) The rate, in litres per minute, at which water leaks out of the tank, which is proportional to the volume in the tank **B1**

(b) In the long term the volume stops changing, so $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 0 \therefore 20 - 0.1V = 0$ **M1**

$V = 200$ litres: the inflow and the leak then balance **A1**

Solve a logistic model and find its limit

7 marks | Grade A*

1. Separate the variables and split the P-side into partial fractions.
2. Integrate, use the initial condition and combine the logs.
3. Rearrange to make P the subject, then let $t \to \infty$

Example. The number of insects, P, in a colony t weeks after it is first observed is modelled by $\dfrac{\mathrm{d}P}{\mathrm{d}t} = \dfrac{P(800-P)}{1600}$, with $P = 100$ when $t = 0$.

(a) Show that $P = \dfrac{800}{1 + 7\mathrm{e}^{-0.5t}}$.

(b) State the long-term population predicted by the model.

Model answer.

(a) $\displaystyle\int \frac{1600}{P(800-P)}\,\mathrm{d}P = \int 1\,\mathrm{d}t$ **M1**

$\dfrac{1600}{P(800-P)} = \dfrac{2}{P} + \dfrac{2}{800-P}$ **M1**

$2\ln P - 2\ln(800-P) = t + c$ **A1**

$t = 0,\ P = 100$: $c = 2\ln\dfrac{100}{700}$

$= 2\ln\dfrac{1}{7}$ **M1**

$2\ln\dfrac{7P}{800-P} = t$

$\Rightarrow \dfrac{7P}{800-P} = \mathrm{e}^{0.5t}$

$\Rightarrow P(7 + \mathrm{e}^{0.5t}) = 800\mathrm{e}^{0.5t}$ **M1**

Dividing by $\mathrm{e}^{0.5t}$: $P = \dfrac{800}{1 + 7\mathrm{e}^{-0.5t}}$ **A1***

(b) As $t \to \infty$, $\mathrm{e}^{-0.5t} \to 0$, so $P \to 800$ insects **B1**

Evaluate a model against data 4 marks **Grade A**

1. Use the model to predict the value at the time of the real data.
2. Compare: does the model overestimate or underestimate, and by how much?
3. Give a specific limitation linked to the context, not a general one.

Example. The number of rabbits, N, on an island t years after 40 rabbits were introduced is modelled by $N = 40\mathrm{e}^{0.3t}$.
(a) After 4 years there were 110 rabbits. Comment on the model's prediction for this time.
(b) Give one reason why the model is unlikely to be suitable in the long term.

Model answer.

(a) $N = 40\mathrm{e}^{1.2}$ **M1** $= 133$ rabbits (3 s.f.) **A1**

The model overestimates the number of rabbits (133 compared with 110), so the growth rate in the model is too high **B1**

(b) The model predicts unlimited growth, but food and space on the island are limited, so the population would level off **B1**

Don't lose marks

- Rounding k early (e.g. 0.14 instead of $\frac{1}{5}\ln 2$), so later answers are wrong at 3 s.f.
- Answering interpretation questions without context: 'it tends to 800' should be '800 insects'.
- Vague limitations such as 'it is not accurate': say what is unrealistic and why.
- Mixing up $t = 0$ with the first year, or giving a time in the wrong units.
- Losing the minus sign: with $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - 20)$, k is positive and the minus sign shows the temperature is falling.

More tips

Memory tricks

- Long term: look for e^{-kt} and replace it with 0.
- The constant term in a cooling model is the temperature of the surroundings: the value the object tends towards.
- Where the rate of change is zero the quantity stops changing: that gives the limiting value without solving.
- Sense-check answers: times positive, populations realistic, temperatures between the starting value and the surroundings.

Exam technique

- Read the context carefully: 'proportional to', 'rate of decrease' and 'initially' all translate into mathematics.
- For 'interpret' or 'explain' marks, one clear sentence using the numbers and units of the context is enough.
- Give final answers to 3 s.f. unless told otherwise, and to a sensible accuracy for the context (e.g. whole rabbits).
- If you cannot do a 'show that' part, you can still use the given result in the later parts.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade C — **Solve exponential growth and decay models**

$\frac{dN}{dt} = kN$ gives $N = N_0e^{kt}$; $k > 0$ means growth and $k < 0$ decay.

Grade C — **Use given data to find the constants**

An initial value gives the constant of integration; a second data point gives k, usually by taking logs.

Grade B — **Find a time or value from a solution**

Substitute into the solution and rearrange, using logs to solve for t

Grade B — **Interpret terms and constants in context**

Say what each term represents, with units, e.g. a constant inflow rate or a rate proportional to the amount present.

Grade A — **Find the long-term (limiting) value**

As $t \to \infty$, $e^{-kt} \to 0$ for $k > 0$; the limit is also where the rate of change is zero.

Grade A — **Evaluate a model and state its limitations**

Compare predictions with data, say why the model may not hold, and suggest a sensible refinement.

Grade A* — **Solve and interpret logistic-type models**

Use partial fractions to solve $\frac{dP}{dt} = kP(M - P)$, then interpret the limiting value M

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