Separable differential equations

P8.7 · Pure mathematics · revision notes

A first order differential equation of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)\mathrm{g}(y)$ is solved by separating the variables: $\int \frac{1}{\mathrm{g}(y)}\,\mathrm{d}y = \int \mathrm{f}(x)\,\mathrm{d}x$. The general solution contains a constant; a particular solution uses a given condition to find it. Questions are 4 to 8 marks and often need partial fractions, substitution or parts along the way.

Key facts

- $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)\mathrm{g}(y) \Rightarrow \int \frac{1}{\mathrm{g}(y)}\,\mathrm{d}y = \int \mathrm{f}(x)\,\mathrm{d}x$
- $\int \frac{1}{y}\,\mathrm{d}y = \ln|y|$
- $\ln|y| = \mathrm{f}(x) + c \Rightarrow y = A\mathrm{e}^{\mathrm{f}(x)}$
- $\mathrm{e}^{a+b} = \mathrm{e}^{a}\mathrm{e}^{b}$
- $\ln a + \ln b = \ln ab$, $k\ln a = \ln a^{k}$
- General solution: includes the constant. Particular solution: constant found from a condition

Notes

Separating the variables

- A differential equation is separable if the right-hand side is a function of x times a function of y: $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)\mathrm{g}(y)$. You may need to factorise to see this.
- Rearrange to $\frac{1}{\mathrm{g}(y)}\,\mathrm{d}y = \mathrm{f}(x)\,\mathrm{d}x$, then integrate both sides: $\int \frac{1}{\mathrm{g}(y)}\,\mathrm{d}y = \int \mathrm{f}(x)\,\mathrm{d}x$
- Divide by the y function: $\frac{\mathrm{d}y}{\mathrm{d}x} = y^{2}\cos x$ becomes $\int y^{-2}\,\mathrm{d}y = \int \cos x\,\mathrm{d}x$

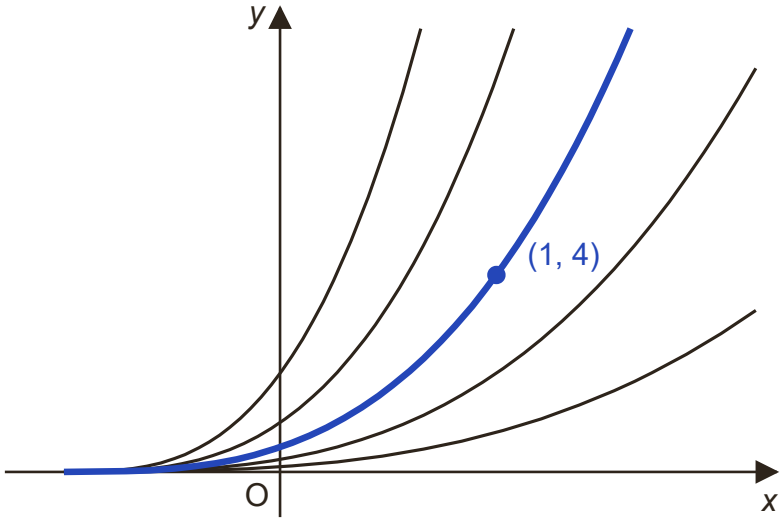
The constant

- One constant is enough: c on the x side covers the constants from both integrals.
- From $\ln|y| = \mathrm{f}(x) + c$ you get $y = \pm\mathrm{e}^{c}\mathrm{e}^{\mathrm{f}(x)} = A\mathrm{e}^{\mathrm{f}(x)}$, where A is a constant. Note that $\mathrm{e}^{\mathrm{f}(x)+c}$ is not $\mathrm{e}^{\mathrm{f}(x)} + c$
- When every term is a log, writing the constant as $\ln k$ lets you combine everything into one log.

Particular solutions

- The **general solution** contains the constant. A **particular solution** uses a condition (e.g. $y = 4$ when $x = 1$) to find it.

general solution: y = A(x + 1)³
particular solution: A = ½



For $(x + 1)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3y$: the general solution is a family of curves; the condition $y = 4$ when $x = 1$ picks out one.

- Substitute the condition straight after integrating, before rearranging: the algebra is easier.
- Then rearrange into the form the question asks for, e.g. $y = \ldots$
- Check by differentiating your answer and substituting it into the original equation.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Find the general solution 4 marks **Grade C**

1. Separate the variables and put an integral sign on each side.
2. Integrate both sides and add one constant.
3. Rearrange to the form asked for, taking care with exponentials.

Example. Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2y$, $y > 0$, giving your answer in the form $y = \mathrm{f}(x)$

Model answer.

$\int \frac{1}{y}\,\mathrm{d}y = \int 6x^2\,\mathrm{d}x$ **M1**

$\ln y = 2x^3 + c$ **A1**

$y = \mathrm{e}^{2x^3+c} = \mathrm{e}^c\mathrm{e}^{2x^3}$ **M1**

$y = A\mathrm{e}^{2x^3}$ **A1**

Find a particular solution 5 marks **Grade B**

1. Separate the variables and integrate.
2. Substitute the condition to find the constant, writing it as a log if the other terms are logs.
3. Combine the logs and remove them to give y in terms of x

Example. Given that $y = 4$ when $x = 1$, solve the differential equation $(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} = 3y$, $x > -1$, $y > 0$, giving y in terms of x

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Model answer.

$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{3}{x+1}\,\mathrm{d}x$ **M1**

$\ln y = 3\ln(x+1) + c$ **A1**

$x = 1,\ y = 4$: $\ln 4 = 3\ln 2 + c$, so $c = \ln 4 - \ln 8 = \ln\frac{1}{2}$ **M1**

$\ln y = \ln\frac{(x+1)^3}{2}$ **M1**

$y = \frac{1}{2}(x+1)^3$ **A1**

Factorise, then separate

5 marks **Grade A**

1. Take out the common factor to write the right-hand side as $\mathbf{f}(x)\mathbf{g}(y)$
2. Separate, integrate and use the condition to find the constant.
3. Rearrange carefully: take the reciprocal of the whole side, not of each term.

Example. Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = xy^2 + y^2$, given that $y = 1$ when $x = 0$. Give y in terms of x

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Model answer.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = y^2(x+1)$ **M1**

$\displaystyle\int y^{-2}\,\mathrm{d}y = \int (x+1)\,\mathrm{d}x \Rightarrow -\frac{1}{y} = \frac{1}{2}x^2 + x + c$ **M1 A1**

$x = 0,\ y = 1 \therefore c = -1$ **M1**

$\dfrac{1}{y} = 1 - x - \dfrac{1}{2}x^2$, so $y = \dfrac{2}{2 - 2x - x^2}$ **A1**

Separate using partial fractions

6 marks | **Grade A***

1. Separate the variables; the y-side is a fraction with a factorising denominator.
2. Split it into partial fractions and integrate both sides.
3. Use the condition, combine the logs, then make y the subject.

Example. Given that $y = 3$ when $x = 1$, solve $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 1$, for $0 < x < \sqrt{2}$, $y > 1$. Give y in terms of x

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Model answer.

$$\int \frac{1}{(y-1)(y+1)}\,\mathrm{d}y = \int \frac{1}{x}\,\mathrm{d}x$$ **M1**

$$\frac{1}{(y-1)(y+1)} = \frac{1}{2(y-1)} - \frac{1}{2(y+1)}$$ **M1**

$$\frac{1}{2}\ln(y-1) - \frac{1}{2}\ln(y+1) = \ln x + c$$ **A1**

$x = 1, y = 3$: $c = \dfrac{1}{2}\ln\dfrac{2}{4}$, so $\ln\dfrac{y-1}{y+1} = 2\ln x + \ln\dfrac{1}{2}$ **M1**

$$\frac{y-1}{y+1} = \frac{x^2}{2} \Rightarrow 2y - 2 = x^2 y + x^2$$ **M1**

$$y = \frac{x^2+2}{2-x^2}$$ **A1**

Don't lose marks

- Integrating with a y still on the x side (or an x on the y side).
- Writing $y = \mathrm{e}^{2x^3} + c$ from $\ln y = 2x^3 + c$
- Taking reciprocals term by term: $\dfrac{1}{y} = 1 - x - \dfrac{1}{2}x^2$ does not give $y = 1 - \dfrac{1}{x} - \dfrac{2}{x^2}$
- Leaving out the constant, or trying to find it before integrating.
- Forgetting the $\dfrac{1}{2}$ factors from the partial fractions on the y side.

More tips

Memory tricks

- 'ys with $\mathrm{d}y$, xs with $\mathrm{d}x$' before you integrate anything.
- Put the condition in as soon as you have integrated: finding c from the log form is easier than from a rearranged formula.
- Write c as $\ln k$ when all the other terms are logs.
- Check: differentiate your solution and substitute it back into the differential equation.

Exam technique

- 'General solution' means leave the constant in; 'particular solution' or a given condition means find it.
- Give the answer in the form asked: 'y in terms of x' needs $y = \ldots$, not an equation still containing logs.
- Show the separated form with an integral sign on each side: it is usually the first method mark.
- In a 'show that', write every step, including the log laws you use to remove the logs.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade C — **Separate the variables correctly**

Get every y on the side with $\mathrm{d}y$ and every x on the side with $\mathrm{d}x$

Grade C — **Find a general solution with one constant**

Integrate both sides and add a single constant, usually on the x side.

Grade B — **Find a particular solution from a condition**

Substitute the given values of x and y to find the constant.

Grade B — **Rearrange a logarithmic solution into y = …**

From $\ln|y| = \mathrm{f}(x) + c$, write $y = A\mathrm{e}^{\mathrm{f}(x)}$, where A is a constant.

Grade A — **Factorise before separating**

For example $\dfrac{\mathrm{d}y}{\mathrm{d}x} = xy^2 + y^2 = y^2(x+1)$: take out the common factor first.

Grade A* — **Separate when integrals need further methods**

The y-side may need partial fractions, e.g. $\displaystyle\int \frac{1}{y^2-1}\,\mathrm{d}y$, and the x-side may need parts or substitution.

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