Integration using partial fractions

P8.6 · Pure mathematics · revision notes

Rational functions whose denominators factorise into linear factors can be split into partial fractions and then integrated term by term, mostly into logarithms. The usual exam form has two parts: (a) write the expression in partial fractions, (b) hence find the integral, often as an exact value in the form $\ln k$ or $a + b\ln c$

Key facts

- $\dfrac{px+q}{(ax+b)(cx+d)} = \dfrac{A}{ax+b} + \dfrac{B}{cx+d}$
- $\dfrac{\text{linear or quadratic}}{(ax+b)(cx+d)^2} = \dfrac{A}{ax+b} + \dfrac{B}{cx+d} + \dfrac{C}{(cx+d)^2}$
- $\displaystyle\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$
- $\displaystyle\int \frac{1}{(ax+b)^2}\,\mathrm{d}x = -\frac{1}{a(ax+b)} + c$
- Improper fraction: divide first
- $\ln a + \ln b = \ln ab$, $\ln a - \ln b = \ln \dfrac{a}{b}$, $k\ln a = \ln a^k$

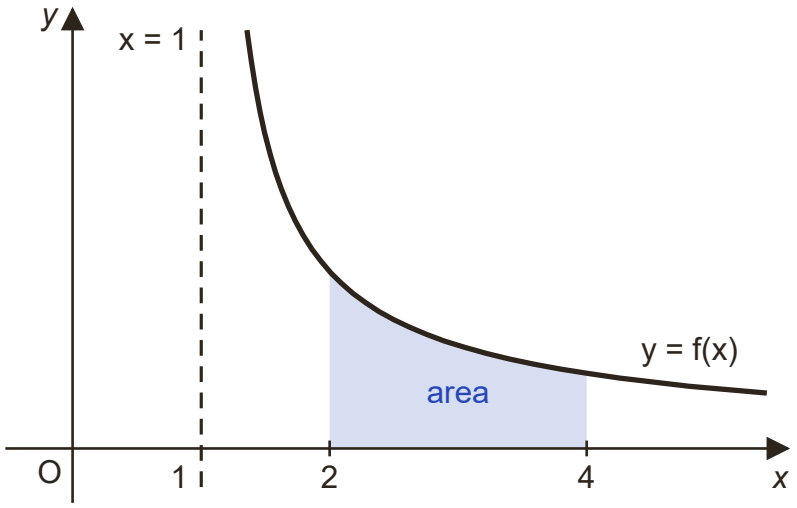
Notes

Splitting into partial fractions

- Distinct linear factors: $\dfrac{5x+1}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$
- A repeated factor needs two terms: $\dfrac{x+5}{(x+1)(x+3)^2} = \dfrac{A}{x+1} + \dfrac{B}{x+3} + \dfrac{C}{(x+3)^2}$
- Multiply through by the denominator, then substitute the value of x that makes each factor zero to find constants quickly; compare coefficients for any that remain.
- Improper fraction (degree of numerator $\geq$ degree of denominator): divide first, e.g. $\dfrac{x^2}{x^2-1} = 1 + \dfrac{1}{x^2-1}$, then split the proper part.

Integrating each term

- $\int \frac{A}{ax+b}\,\mathrm{d}x = \frac{A}{a}\ln|ax+b| + c$. Watch the sign when the x term inside is negative:
  $\int \frac{1}{3-x}\,\mathrm{d}x = -\ln|3-x| + c$



$\int_2^4 \left(\frac{2}{x-1} + \frac{3}{x+2}\right)\mathrm{d}x = \ln\frac{243}{8}$ is the shaded area; the limits must stay to one side of the asymptote.

$$\int \frac{B}{(ax+b)^2}\,\mathrm{d}x = \int B(ax+b)^{-2}\,\mathrm{d}x$$
$$= -\frac{B}{a}(ax+b)^{-1} + c$$

- Polynomial terms from the division integrate as powers of x

Exact answers

- Combine logs with the laws of logarithms:

$$2\ln 3 + 3\ln 6 - 3\ln 4 = \ln 9 + \ln\frac{27}{8}$$
$$= \ln\frac{243}{8}$$

- Keep the modulus signs when a bracket is negative at a limit: $\ln|x-3|$ at $x=1$ is $\ln 2$

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, dM = depends on the earlier method mark, A1* = the solution must be fully correct, with every step shown, ft = follow through: you still earn the mark if you use your own earlier wrong answer correctly.

Partial fractions, then integrate (distinct factors) 7 marks **Grade C**

1. Write the identity for the numerator and substitute the roots of each factor to find the constants.
2. Integrate each term to a logarithm, dividing by the coefficient of x where needed.
3. Substitute the limits and combine the logs into the form asked for.

Example. $\mathrm{f}(x) = \dfrac{5x+1}{(x-1)(x+2)}$, $x > 1$.

(a) Express $\mathrm{f}(x)$ in partial fractions.

(b) Hence find the exact value of $\displaystyle\int_2^4 \mathrm{f}(x)\,\mathrm{d}x$, giving your answer in the form $\ln k$, where k is a rational number.

Model answer.

(a) $5x+1 = A(x+2)+B(x-1)$ **M1**

$x=1$: $6=3A$, so $A=2$ **A1**; $x=-2$: $-9=-3B$, so $B=3$ **A1**

$$\mathrm{f}(x) = \frac{2}{x-1} + \frac{3}{x+2}$$

(b) $[2\ln(x-1)+3\ln(x+2)]_2^4$ **M1 A1ft**

$= (2\ln 3 + 3\ln 6)$
$\quad -(2\ln 1 + 3\ln 4)$

$= \ln 9 + \ln \dfrac{27}{8}$ **M1**

$= \ln \dfrac{243}{8}$ **A1**

Integrate with a repeated linear factor

9 marks **Grade B**

1. Use three terms, including one over the squared factor.
2. Find two constants by substitution and the third by comparing coefficients.
3. Integrate: the single factors give logs, the squared factor gives a negative power.
4. Substitute the limits and simplify.

Example. (a) Express $\dfrac{x+5}{(x+1)(x+3)^2}$ in partial fractions.

(b) Hence show that $\displaystyle\int_0^1 \frac{x+5}{(x+1)(x+3)^2}\,\mathrm{d}x = \ln\frac{3}{2} - \frac{1}{12}$

Model answer.

(a) $x+5$

$= A(x+3)^2$

$+ B(x+1)(x+3)$

$+ C(x+1)$ **M1**

$x=-1$: $4=4A$, so $A=1$; $x=-3$: $2=-2C$, so $C=-1$ **A1**

Comparing x^2 terms: $0=A+B$, so $B=-1$ **M1 A1**

$\dfrac{1}{x+1} - \dfrac{1}{x+3} - \dfrac{1}{(x+3)^2}$

(b) $\left[\ln(x+1) - \ln(x+3) + \dfrac{1}{x+3}\right]_0^1$ **M1** (logs) **M1** ($k(x+3)^{-1}$) **A1**

$= \left(\ln 2 - \ln 4 + \dfrac{1}{4}\right) - \left(0 - \ln 3 + \dfrac{1}{3}\right)$ **dM1**

$= \ln\dfrac{2\times 3}{4} + \dfrac{1}{4} - \dfrac{1}{3}$

$= \ln\dfrac{3}{2} - \dfrac{1}{12}$ **A1***

Divide first, then use partial fractions

6 marks **Grade A**

1. Check the degrees: if the numerator's degree is at least the denominator's, divide.
2. Split the proper fraction that is left into partial fractions.
3. Integrate every term and simplify the logs to reach the given form.

Example. Show that $\displaystyle\int_2^3 \frac{x^2}{x^2-1}\,\mathrm{d}x = 1 + \frac{1}{2}\ln\frac{3}{2}$

Model answer.

$$\frac{x^2}{x^2-1} = 1 + \frac{1}{x^2-1}$$ **M1**

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \text{ with } A = \frac{1}{2},\ B = -\frac{1}{2}$$ **M1 A1**

$$\text{Integral} = x + \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)$$ **M1**

$$\left(3 + \frac{1}{2}\ln 2 - \frac{1}{2}\ln 4\right) - \left(2 + \frac{1}{2}\ln 1 - \frac{1}{2}\ln 3\right)$$

$$= 1 + \frac{1}{2}\ln\frac{2\times 3}{4}$$ **dM1**

$$= 1 + \frac{1}{2}\ln\frac{3}{2}$$ **A1***

Don't lose marks

- Forgetting the $\frac{1}{a}$: $\displaystyle\int \frac{1}{2x+1}\,\mathrm{d}x = \frac{1}{2}\ln|2x+1| + c$
- Integrating $\dfrac{1}{(x+3)^2}$ to a logarithm.
- Missing the minus sign in $\displaystyle\int \frac{1}{3-x}\,\mathrm{d}x = -\ln|3-x| + c$
- Splitting an improper fraction without dividing first, so the constants cannot be found.
- Sign slips at the lower limit: put the whole lower-limit value in brackets before subtracting.

More tips

Memory tricks

- Cover-up rule for a distinct factor $(x - p)$: cover it up in the original fraction and put $x = p$ into what is left to get its constant.
- Check your partial fractions by putting $x = 0$ (or any easy value) into both forms.
- Linear denominator gives $\ln$; squared denominator gives a negative power.
- Before splitting, compare degrees: top $\geq$ bottom means divide first.

Exam technique

- 'Hence' means use your partial fractions from part (a); even if part (a) is wrong you can still earn method marks in part (b).
- Give the answer in exactly the form asked (e.g. $\ln k$, or $a + b \ln c$), stating the constants.
- Show the identity and the values you substitute: these earn method marks even if you make an arithmetic slip.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade C — **Split into partial fractions with distinct factors**

Write $\dfrac{px + q}{(ax + b)(cx + d)} = \dfrac{A}{ax + b} + \dfrac{B}{cx + d}$ and find A and B by substituting values or comparing coefficients.

Grade C — **Integrate A/(ax + b) to a logarithm**

$$\int \frac{A}{ax + b}\,\mathrm{d}x = \frac{A}{a}\ln|ax + b| + c$$

Grade B — **Integrate a repeated-factor term**

$\displaystyle\int \frac{B}{(ax + b)^2}\,\mathrm{d}x = -\frac{B}{a(ax + b)} + c$: a power, not a logarithm.

Grade B — **Combine logarithms into an exact answer**

Use the laws of logarithms to write the answer in the form the question asks for.

Grade A — **Divide first for improper fractions**

If the degree of the numerator is at least that of the denominator, divide to get a polynomial plus a proper fraction.

Grade A* — **Use partial fractions inside longer problems**

Partial fractions often appear after a substitution or when separating the variables in a differential equation.

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