# Integration by parts

P8.5b · Pure mathematics · revision notes

Integration by parts reverses the product rule: $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$ (in the formula booklet). Use it for products such as $x\mathrm{e}^x$, $x\sin x$ and $x^2\ln x$, and for $\ln x$ itself. Questions are usually 4 to 7 marks, often a definite integral with an exact answer, and you may need to apply the method twice.

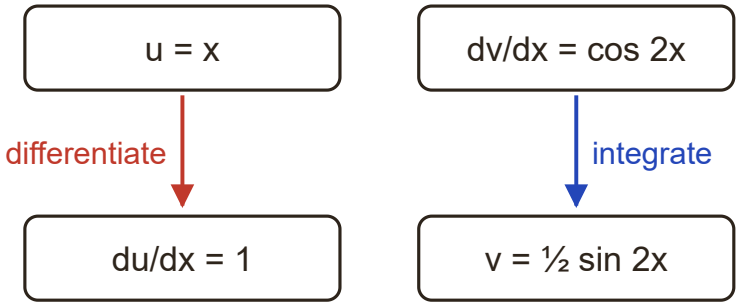
## Key facts

- $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$
- Choosing $u$: LATE (logs, algebraic, trig, exponentials)
- $\int x\mathrm{e}^x\,\mathrm{d}x = x\mathrm{e}^x - \mathrm{e}^x + c$
- $\int x\sin x\,\mathrm{d}x = -x\cos x + \sin x + c$
- $\int \ln x\,\mathrm{d}x = x\ln x - x + c$
- $\int x\cos x\,\mathrm{d}x = x\sin x + \cos x + c$
- Definite: $[uv]_a^b - \int_a^b v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

## Notes

### The formula

- $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$. It comes from integrating the product rule, $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$
- Write out the four pieces, $u$, $\frac{\mathrm{d}u}{\mathrm{d}x}$, $\frac{\mathrm{d}v}{\mathrm{d}x}$ and $v$, before you substitute into the formula.




Fill in the four boxes first, then substitute into $uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$ (shown for $\int x\cos 2x\,\mathrm{d}x$).

## Choosing u

- The new integral must be easier than the old one. Choose $u$ in the order **L**ogs, **A**lgebraic (powers of $x$), **T**rig, **E**xponentials: the first type in this list that is present is $u$
- If $\ln x$ is present it must be $u$, because you cannot easily integrate it. For $\int \ln x \, \mathrm{d}x$, take $u = \ln x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$: $\int \ln x \, \mathrm{d}x = x \ln x - x + c$
- Otherwise take $u$ as the power of $x$: each application lowers the power by one.

## Harder cases

- Twice: $\int x^2 \mathrm{e}^x \, \mathrm{d}x = x^2 \mathrm{e}^x - \int 2x \mathrm{e}^x \, \mathrm{d}x$, and the second integral needs parts again. Keep it in brackets to get the signs right.
- Definite: $\int_a^b u \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x = [uv]_a^b - \int_a^b v \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x$. Useful values: $\ln 1 = 0$, $\ln \mathrm{e} = 1$, $\sin \pi = 0$, $\cos \pi = -1$
- If the original integral $I$ reappears after two applications (as with $\mathrm{e}^x \sin x$), write the result as an equation in $I$ and solve it. Grade A* only
- Reduction formulae are not required.

## How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, dM = depends on the earlier method mark, A1* = the solution must be fully correct, with every step shown.

**Integrate x times a trigonometric or exponential function** 4 marks Grade C

1. Let $u = x$ and let $\frac{\mathrm{d}v}{\mathrm{d}x}$ be the trigonometric or exponential part.
2. Find $v$, remembering to divide by the coefficient of $x$
3. Substitute into the formula, integrate the remaining standard integral and add $+c$

**Example.** Find $\int x \cos 2x \, \mathrm{d}x$

**Model answer.**

$u = x$, $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos 2x$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $v = \frac{1}{2}\sin 2x$

$\int x \cos 2x \, \mathrm{d}x = \frac{1}{2} x \sin 2x - \int \frac{1}{2} \sin 2x \, \mathrm{d}x$ **M1 A1**

$= \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + c$ **dM1 A1**

**Integrate a product involving ln x**

5 marks **Grade B**

1. Let $u = \ln x$, so $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{x}$
2. Integrate the other factor to get $v$ and apply the formula; the $\dfrac{1}{x}$ simplifies the new integral.
3. Apply the limits to both parts, using $\ln 1 = 0$ and $\ln \mathrm{e} = 1$

**Example.** Show that $\displaystyle\int_1^{\mathrm{e}} x^2 \ln x \,\mathrm{d}x = \frac{2\mathrm{e}^3 + 1}{9}$

---

**Model answer.**

$u = \ln x$, $\dfrac{\mathrm{d}v}{\mathrm{d}x} = x^2$, so $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{x}$ and $v = \dfrac{x^3}{3}$

$\left[\dfrac{x^3}{3}\ln x\right]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} \dfrac{x^3}{3} \times \dfrac{1}{x}\,\mathrm{d}x$ **M1 A1**

$= \dfrac{\mathrm{e}^3}{3} - \left[\dfrac{x^3}{9}\right]_1^{\mathrm{e}}$ **M1**

$= \dfrac{\mathrm{e}^3}{3} - \left(\dfrac{\mathrm{e}^3}{9} - \dfrac{1}{9}\right)$ **dM1**

$= \dfrac{2\mathrm{e}^3 + 1}{9}$ **A1***

**Apply integration by parts twice**

6 marks **Grade A**

1. Let $u = x^2$ and apply the formula once.
2. The new integral is $x$ times a function: apply parts again, keeping it in brackets.
3. Collect the terms, then apply the limits.

**Example.** Find the exact value of $\displaystyle\int_0^1 x^2\mathrm{e}^x \,\mathrm{d}x$

---

**Model answer.**

$u = x^2$, $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$: $\int x^2\mathrm{e}^x \,\mathrm{d}x = x^2\mathrm{e}^x - \int 2x\mathrm{e}^x \,\mathrm{d}x$ **M1 A1**

Again, with $u = 2x$: $\int 2x\mathrm{e}^x \,\mathrm{d}x = 2x\mathrm{e}^x - \int 2\mathrm{e}^x \,\mathrm{d}x = 2x\mathrm{e}^x - 2\mathrm{e}^x$ **M1**

So $\int x^2\mathrm{e}^x \,\mathrm{d}x = x^2\mathrm{e}^x - 2x\mathrm{e}^x + 2\mathrm{e}^x + c$ **A1**

$\left[\mathrm{e}^x(x^2 - 2x + 2)\right]_0^1$

$= \mathrm{e}(1) - 1(2)$ **dM1**

$= \mathrm{e} - 2$ **A1**

**Integrals where the original integral reappears** 6 marks **Grade A***

1. Apply parts once, then again to the new integral, choosing the same type of function for $u$ both times (here the trigonometric one).
2. Call the original integral $I$: it reappears on the right-hand side.
3. Collect the $I$ terms, divide, then apply the limits.

**Example.** Show that $\displaystyle\int_0^{\pi} \mathrm{e}^x \sin x \,\mathrm{d}x = \frac{1}{2}(\mathrm{e}^{\pi}+1)$

**Model answer.**

Let $I = \int \mathrm{e}^x \sin x \,\mathrm{d}x$. With $u = \sin x$, $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$: $I = \mathrm{e}^x \sin x - \int \mathrm{e}^x \cos x \,\mathrm{d}x$ **M1**

With $u = \cos x$: $\int \mathrm{e}^x \cos x \,\mathrm{d}x = \mathrm{e}^x \cos x + \int \mathrm{e}^x \sin x \,\mathrm{d}x = \mathrm{e}^x \cos x + I$ **M1**

So $I = \mathrm{e}^x \sin x - \mathrm{e}^x \cos x - I$ **A1**

$2I = \mathrm{e}^x(\sin x - \cos x)$, so $I = \dfrac{1}{2}\mathrm{e}^x(\sin x - \cos x) + c$ **M1 A1**

$$\left[\frac{1}{2}\mathrm{e}^x(\sin x - \cos x)\right]_0^{\pi}$$

$$= \frac{1}{2}\mathrm{e}^{\pi}(0+1) - \frac{1}{2}(0-1)$$

$$= \frac{1}{2}(\mathrm{e}^{\pi}+1)$$ **A1***

## Don't lose marks

- Choosing $u = \mathrm{e}^x$ in $\int x\mathrm{e}^x \,\mathrm{d}x$, which makes the new integral harder.
- Sign errors in the second application: put it in brackets, $x^2\mathrm{e}^x - (2x\mathrm{e}^x - 2\mathrm{e}^x)$
- Forgetting to divide by $k$ when finding $v$: if $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \cos 2x$, then $v = \dfrac{1}{2}\sin 2x$
- Applying the limits to the integral part but not to $uv$
- Trying to use $\ln x$ as $\dfrac{\mathrm{d}v}{\mathrm{d}x}$: it must be $u$

## More tips

### Memory tricks

- LATE decides $u$: Logs, Algebraic, Trig, Exponentials.
- Say it as you write it: '$uv$ minus the integral of $v$ times $\frac{\mathrm{d}u}{\mathrm{d}x}$'.
- Check by differentiating with the product rule: you should get back the integrand.
- If the new integral looks harder than the old one, swap your choices of $u$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$

### Exam technique

- Write down $u$, $\frac{\mathrm{d}u}{\mathrm{d}x}$, $\frac{\mathrm{d}v}{\mathrm{d}x}$ and $v$ clearly: the correct formula with your values earns the first method mark.
- The formula is in the booklet, but practise until choosing $u$ is automatic.
- When a later part says 'Hence', for example to find an area, use your result rather than starting again.
- Leave exact answers in terms of $\mathrm{e}$, $\ln$ and $\pi$, simplified.

## What each grade needs

What you need to be able to do, from the first marks up to the top grade.

**Grade C** — **Choose u and dv/dx, then apply the formula**

Pick $u$ so that $\frac{\mathrm{d}u}{\mathrm{d}x}$ is simpler, and $\frac{\mathrm{d}v}{\mathrm{d}x}$ so that it is easy to integrate.

**Grade C** — **Integrate x times an exponential or trig function**

Let $u = x$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and the new integral is a standard one.

**Grade B** — **Integrate products involving ln x**

Let $u = \ln x$; for $\int \ln x \, \mathrm{d}x$, write it as $\int 1 \times \ln x \, \mathrm{d}x$

**Grade B** — **Evaluate definite integrals by parts exactly**

Apply the limits to both $uv$ and the remaining integral.

**Grade A** — **Apply integration by parts twice**

For $x^2\mathrm{e}^x$ or $x^2 \sin x$, the first application leaves $x$ times a function, so apply the formula again.

**Grade A*** — **Solve integrals where the original reappears**

For $\int \mathrm{e}^x \sin x \, \mathrm{d}x$, two applications bring back the original integral $I$; rearrange to find $I$

Free notes, Full packs, practice papers and mock exams at chhetriacademy.com. Written for this site; not taken from past papers. Stuck? 1-to-1 tutoring with a free 30-minute trial lesson.