Integration by substitution

P8.5a · Pure mathematics · revision notes

Integration by substitution reverses the chain rule: you change the variable from x to a new variable u, replace $\mathrm{d}x$, change the limits and integrate in u. Closely linked are integrals you can do 'by inspection', such as $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$. Expect 5 to 8 marks, usually with the substitution given.

Key facts

- $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$
- $\int \mathrm{f}'(x)[\mathrm{f}(x)]^n\,\mathrm{d}x = \frac{[\mathrm{f}(x)]^{n+1}}{n+1} + c$, $n \neq -1$
- $\int \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}\,\mathrm{d}x = \mathrm{e}^{\mathrm{f}(x)} + c$
- Replace $\mathrm{d}x$ using $\frac{\mathrm{d}u}{\mathrm{d}x}$, e.g. $\mathrm{d}x = \frac{\mathrm{d}u}{2x}$
- $x = a\sin\theta$: $\sqrt{a^2 - x^2} = a\cos\theta$, $\mathrm{d}x = a\cos\theta\,\mathrm{d}\theta$
- Definite integrals: change the limits to u-values
- $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

Notes

Integrating by inspection

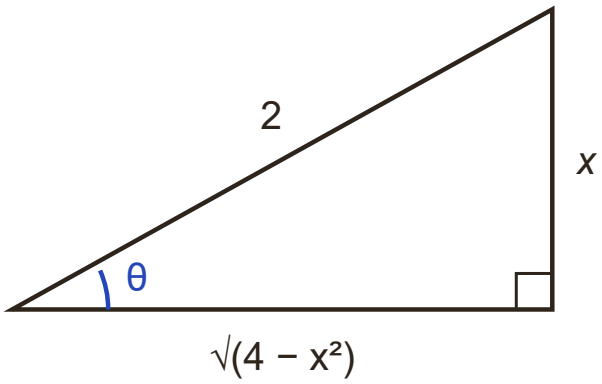
- $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$. For example, $\int \frac{3x^2}{x^3+1}\,\mathrm{d}x = \ln|x^3+1| + c$, and $\int \tan x\,\mathrm{d}x = -\ln|\cos x| + c = \ln|\sec x| + c$
- $\int \mathrm{f}'(x)[\mathrm{f}(x)]^n\,\mathrm{d}x = \frac{[\mathrm{f}(x)]^{n+1}}{n+1} + c$ and $\int \mathrm{f}'(x)\mathrm{e}^{\mathrm{f}(x)}\,\mathrm{d}x = \mathrm{e}^{\mathrm{f}(x)} + c$
- If the derivative is there apart from a constant factor, guess the answer, differentiate it, and adjust the constant.

The method of substitution

- Given $u = g(x)$, find $\frac{du}{dx}$ and rearrange to replace dx, e.g. $u = x^2 + 1 \Rightarrow dx = \frac{du}{2x}$
- Replace every x: write any leftover x in terms of u (e.g. $u = x + 1 \Rightarrow x = u - 1$). The integral must contain only u
- Simplify, then integrate with respect to u
- Indefinite integral: substitute back to give the answer in terms of x. Definite integral: change the limits to u-values and do not substitute back.

Choosing a substitution Grade A or above only

- Try u = the expression inside a root, a power or an exponent, or the denominator. For $\sqrt{ax+b}$, $u = \sqrt{ax+b}$ often works well: then $u^2 = ax + b$ and $2u\,du = a\,dx$
- For $\sqrt{a^2 - x^2}$, try $x = a\sin\theta$, so $\sqrt{a^2 - x^2} = a\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. The new limits are angles in radians.



With $x = 2\sin\theta$: $\sin\theta = \frac{x}{2}$, and the third side is $\sqrt{4 - x^2} = 2\cos\theta$

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, A1* = the solution must be fully correct, with every step shown.

Integrate by inspection (reverse chain rule)

2 marks each | **Grade C**

1. Look for a function whose derivative (up to a constant) is also in the integrand.
2. Write down the form of the answer: $k\ln|\mathrm{f}(x)|$ or $k[\mathrm{f}(x)]^{n+1}$
3. Differentiate your guess and choose k to match; add $+c$

Example. Find (a) $\displaystyle\int \frac{4x}{x^2+3}\,\mathrm{d}x$, (b) $\displaystyle\int x^2(x^3+2)^5\,\mathrm{d}x$

Model answer.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2+3) = 2x$, so the answer has the form $k\ln(x^2+3)$ **M1**;

$$\int \frac{4x}{x^2+3}\,\mathrm{d}x = 2\ln(x^2+3)+c$$ **A1**

(b) Try $(x^3+2)^6$: its derivative is $18x^2(x^3+2)^5$ **M1**, so

$$\int x^2(x^3+2)^5\,\mathrm{d}x = \frac{1}{18}(x^3+2)^6+c$$ **A1**

Use a given substitution for a definite integral

6 marks | **Grade B**

1. Find $\mathrm{d}x$ in terms of $\mathrm{d}u$, and any leftover x in terms of u
2. Change both limits to u-values.
3. Write the whole integral in u, simplify into powers of u, then integrate and substitute the u-limits.

Example. Use the substitution $u = x+1$ to find the exact value of $\displaystyle\int_0^3 x\sqrt{x+1}\,\mathrm{d}x$

Model answer.

$x = u-1$ and $\mathrm{d}x = \mathrm{d}u$ **B1**

$x = 0 \Rightarrow u = 1$, $x = 3 \Rightarrow u = 4$ **B1**

$\displaystyle\int_1^4 (u-1)u^{\frac{1}{2}}\,\mathrm{d}u = \int_1^4 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)\mathrm{d}u$ **M1**

$\displaystyle= \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^4$ **M1 A1**

$\displaystyle= \left(\frac{64}{5} - \frac{16}{3}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)$

$\displaystyle= \frac{112}{15} + \frac{4}{15}$

$\displaystyle= \frac{116}{15}$ **A1**

Use a substitution for an indefinite integral

5 marks | Grade A

1. Differentiate the substitution and rearrange to replace $\mathrm{d}x$
2. Write every remaining x (here e^x) in terms of u, and simplify.
3. Integrate in u, then substitute back to give the answer in terms of x, with $+c$

Example. Use the substitution $u = \mathrm{e}^x + 1$ to find $\displaystyle\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^x + 1}\,\mathrm{d}x$

Model answer.

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$, so $\mathrm{d}x = \dfrac{\mathrm{d}u}{\mathrm{e}^x}$, and $\mathrm{e}^x = u - 1$ **M1**

$\displaystyle\int \frac{\mathrm{e}^x \times \mathrm{e}^x}{u} \times \frac{\mathrm{d}u}{\mathrm{e}^x} = \int \frac{u-1}{u}\,\mathrm{d}u$ **A1**

$\displaystyle= \int \left(1 - \frac{1}{u}\right)\mathrm{d}u = u - \ln u + c$ **M1 A1**

$= \mathrm{e}^x + 1 - \ln(\mathrm{e}^x + 1) + c$ **A1**

(the 1 can be absorbed into c)

Use a trigonometric substitution

6 marks | Grade A*

1. Replace $\mathrm{d}x$ and simplify the root with $1 - \sin^2\theta = \cos^2\theta$
2. Change the limits to angles in radians.
3. Use a double angle formula for $\cos^2\theta$ or $\sin^2\theta$, integrate and use exact values.

Example. Use the substitution $x = 2\sin\theta$ to show that $\displaystyle\int_0^{\sqrt{3}} \sqrt{4 - x^2}\,\mathrm{d}x = \frac{2\pi}{3} + \frac{\sqrt{3}}{2}$

Model answer.

$\mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$ **B1**

$\sqrt{4 - 4\sin^2\theta} = 2\cos\theta$, so the integrand becomes $2\cos\theta \times 2\cos\theta = 4\cos^2\theta$ **M1**

$x = 0 \Rightarrow \theta = 0$; $x = \sqrt{3} \Rightarrow \sin\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = \dfrac{\pi}{3}$ **B1**

$\displaystyle\int_0^{\frac{\pi}{3}} 4\cos^2\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{3}} (2 + 2\cos 2\theta)\,\mathrm{d}\theta$ **M1**

$= [2\theta + \sin 2\theta]_0^{\frac{\pi}{3}}$ **M1**

$= \dfrac{2\pi}{3} + \sin\dfrac{2\pi}{3}$

$= \dfrac{2\pi}{3} + \dfrac{\sqrt{3}}{2}$ **A1***

Don’t lose marks

- Replacing the function but not $\mathrm{d}x$, so the integral still mixes x and u
- Keeping the x-limits after changing to u, or substituting back to x and then using the u-limits.
- Losing a constant factor when adjusting, e.g. $\frac{1}{6}(x^3+2)^6$ instead of $\frac{1}{18}(x^3+2)^6$
- Forgetting $+c$, or leaving an indefinite integral in terms of u
- Using degrees for the new limits in a trigonometric substitution.

More tips

Memory tricks

- Spot the derivative: if the derivative of one part of the integrand is also there (up to a constant), integrate by inspection or let u be that part.
- Differentiate your final answer to check it.
- Write the new limits next to the integral sign straight away so you cannot forget them.
- If an x is left over after replacing $\mathrm{d}x$, rearrange the substitution to write x in terms of u

Exam technique

- If the question says 'use the substitution…', you must use it: another method may score nothing.
- Show the line where the integral is written entirely in u with the new limits: it carries method marks.
- Simplify the u-integrand before integrating: most become powers of u, $\frac{1}{u}$ or a standard trigonometric function.
- For a given answer, keep exact values throughout and write out the final substitution of the limits.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade C — **Integrate f′(x)/f(x) by inspection**

When the numerator is a multiple of the derivative of the denominator, the answer is a multiple of $\ln|\mathrm{f}(x)|$

Grade C — **Integrate f′(x)[f(x)]ⁿ by inspection**

$\int \mathrm{f}'(x)[\mathrm{f}(x)]^n\,\mathrm{d}x = \frac{[\mathrm{f}(x)]^{n+1}}{n+1} + c$ for $n \neq -1$; adjust the constant if only a multiple of the derivative is there.

Grade B — **Use a given substitution to integrate**

Replace every x and $\mathrm{d}x$ so that the integral is entirely in u, then integrate.

Grade B — **Change the limits for a definite integral**

Convert each x-limit to a u-limit so that you never need to go back to x

Grade A — **Choose a suitable substitution yourself**

Let u be the awkward part: the inside of a root, a bracket or an exponent, or the denominator.

Grade A* — **Use trigonometric substitutions with identities**

For example $x = 2\sin\theta$ turns $\sqrt{4-x^2}$ into $2\cos\theta$; then use a double angle formula.

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