Areas between curves and parametric areas

P8.3b · Pure mathematics · revision notes

The area between two graphs is $\int_a^b (y_{\text{top}} - y_{\text{bottom}})\,\mathrm{d}x$ between their intersection points. For a curve given parametrically, the area under it is $\int y \dfrac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$, with limits in t.

These are usually 6 to 10 mark questions with a diagram, combining intersections, tangents or normals and integration.

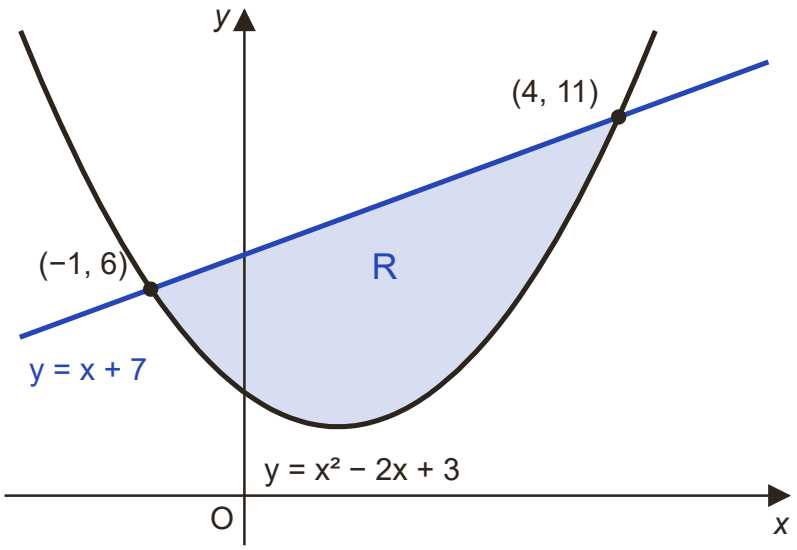
Key facts

- Area between graphs $= \int_a^b (y_{\text{top}} - y_{\text{bottom}})\,\mathrm{d}x$
- Triangle $= \dfrac{1}{2} \times \text{base} \times \text{height}$; trapezium $= \dfrac{1}{2}(a+b)h$
- Normal: gradient $-\dfrac{1}{m}$
- $\sin 2t = 2\sin t \cos t$, $\cos 2t = 1 - 2\sin^2 t = 2\cos^2 t - 1$
- Limits a and b: the x-coordinates of the intersection points
- Tangent: $y - y_1 = m(x - x_1)$ with $m = \dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point
- Grade A or above only Parametric area $= \int_{t_1}^{t_2} y \dfrac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$

Notes

Area between two graphs

- Find the intersection points by solving the equations simultaneously: their x-coordinates are the limits.
- Area $= \int_a^b (y_{\text{top}} - y_{\text{bottom}})\,\mathrm{d}x$. This works wherever the region is, even below the x-axis, as long as the top graph stays above the bottom one for the whole interval.



Area $= \int_{-1}^{4} \left((x+7) - (x^2 - 2x + 3)\right)\,\mathrm{d}x$: top minus bottom, between the intersections.

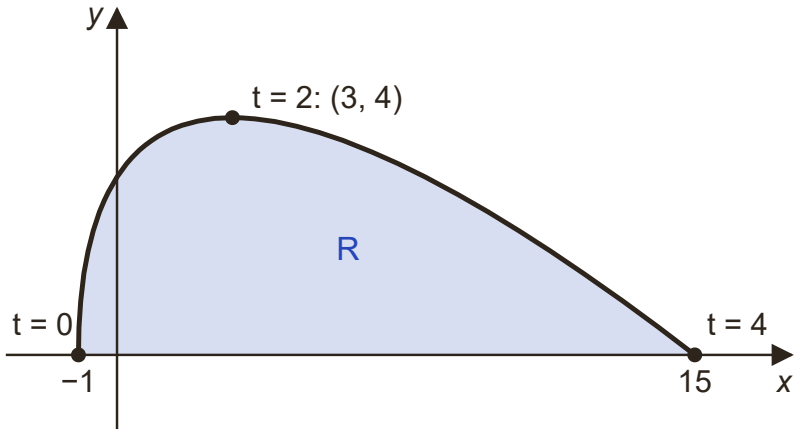
- If the graphs cross inside the interval, split at the crossing point and swap top and bottom.
- Subtract before integrating: one integral of the difference is quicker than two separate integrals.

Combining shapes

- When a straight line is involved, it can be quicker to find the area under the curve and add or subtract a triangle ($\frac{1}{2}bh$) or trapezium.
- For a tangent or normal: differentiate to find the gradient at the point, use $y - y_1 = m(x - x_1)$, and find where the line meets the axis. Label every key point on a sketch before deciding which areas to add or subtract.

Area under a parametric curve Grade A or above only

- If $x = f(t)$ and $y = g(t)$, the area between the curve and the x-axis is $\int y\,dx = \int_{t_1}^{t_2} y\frac{dx}{dt}\,dt$
- Change the limits: find the values of t that give the x-values at the ends of the region.



The x-values -1 and 15 come from $t = 0$ and $t = 4$, so the area is $\int_0^4 y\frac{dx}{dt}\,dt$

- Multiply out and simplify $y\frac{dx}{dt}$, using identities if needed, before integrating.
- For a curve above the axis, a negative answer means x decreases as t increases over the interval; the area is the positive value.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, dM = depends on the earlier method mark, A1* = the solution must be fully correct, with every step shown.

Find the area between a curve and a line 5 marks **Grade C**

1. Solve the equations simultaneously to find the x-coordinates where they meet.
2. Decide which graph is on top between these points (use the sketch or test a value).
3. Integrate (top − bottom) between the limits.

Example. The curve C has equation $y = x^2 - 2x + 3$ and the line l has equation $y = x + 7$.
The finite region R is bounded by C and l.
Find the exact area of R

Model answer.

$x^2 - 2x + 3 = x + 7 \Rightarrow x^2 - 3x - 4 = 0$ **M1**

$(x - 4)(x + 1) = 0$, so $x = -1$ and $x = 4$ **A1**

The line is above the curve between these points, so area

$= \int_{-1}^{4} \left((x + 7) - (x^2 - 2x + 3)\right) \mathrm{d}x$

$= \int_{-1}^{4} (-x^2 + 3x + 4)\, \mathrm{d}x$ **M1**

$= \left[-\frac{x^3}{3} + \frac{3x^2}{2} + 4x\right]_{-1}^{4}$

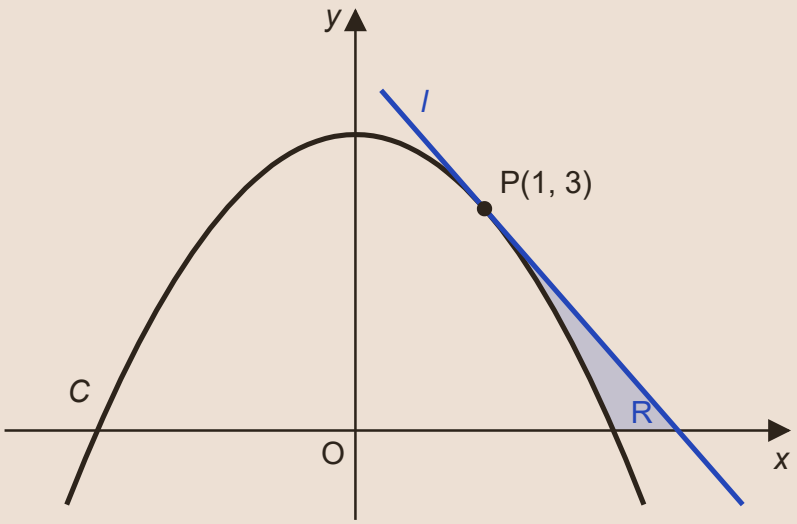
$= \frac{56}{3} - \left(-\frac{13}{6}\right)$ **dM1**

$= \frac{125}{6}$ **A1**

Find the area of a region bounded by a curve, a tangent and an axis 7 marks **Grade B**

1. Differentiate to find the gradient at the point and write down the tangent (or normal).
2. Find where the line and the curve meet the x-axis, and mark every point on a sketch.
3. Split the region into an area under the line (often a triangle) and an area under the curve.
4. Add or subtract as the sketch shows.

Example. The curve C has equation $y = 4 - x^2$. The line l is the tangent to C at the point $P(1, 3)$. The finite region R is bounded by C, l and the positive x-axis.
Find the exact area of R.



Model answer.

$\frac{dy}{dx} = -2x = -2$ at $x = 1$ **M1**

l: $y - 3 = -2(x - 1)$, so $y = 5 - 2x$ **A1**

l meets the x-axis at $x = \frac{5}{2}$ and C meets it at $x = 2$ **B1**

Area under l from $x = 1$ to $x = \frac{5}{2}$ is a triangle: $\frac{1}{2} \times \frac{3}{2} \times 3 = \frac{9}{4}$ **M1**

Area under C from $x = 1$ to $x = 2$: $\left[4x - \frac{x^3}{3}\right]_1^2 = \frac{16}{3} - \frac{11}{3} = \frac{5}{3}$ **M1 A1**

Area of R

$= \frac{9}{4} - \frac{5}{3}$

$= \frac{7}{12}$ **A1**

Find the area under a parametric curve

5 marks **Grade A**

1. Find $\dfrac{\mathrm{d}x}{\mathrm{d}t}$
2. Find the t-values at the ends of the region (e.g. where $y = 0$).
3. Set up $\int y \dfrac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$, multiply out and integrate with respect to t

Example. The curve C has parametric equations $x = t^2 - 1$, $y = 4t - t^2$, $0 \leq t \leq 4$. The region R is bounded by C and the x-axis.
Find the exact area of R

Model answer.

$\dfrac{\mathrm{d}x}{\mathrm{d}t} = 2t$ **B1**

$y = 0$ when $t = 0$ and $t = 4$ **B1**

Area $= \int_0^4 (4t - t^2)(2t)\,\mathrm{d}t = \int_0^4 (8t^2 - 2t^3)\,\mathrm{d}t$ **M1**

$= \left[\dfrac{8t^3}{3} - \dfrac{t^4}{2}\right]_0^4$

$= \dfrac{512}{3} - 128$ **M1**

$= \dfrac{128}{3}$ **A1**

Show that a parametric area has an exact value

6 marks | Grade A*

1. Write the area as $\int y \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$ and change the limits to t
2. Use an identity to simplify the integrand.
3. Integrate, using standard results from the formula booklet, and reach the given answer.

Example. The curve C has parametric equations $x = \tan t$, $y = \sin 2t$, $0 \leq t < \frac{\pi}{2}$. The region R is bounded by C, the x-axis and the line $x = 1$.
Show that the area of R is $\ln 2$

Model answer.

$\frac{\mathrm{d}x}{\mathrm{d}t} = \sec^2 t$ **B1**

$x = 0 \Rightarrow t = 0$ and $x = 1 \Rightarrow t = \frac{\pi}{4}$ **B1**

Area $= \int_0^{\frac{\pi}{4}} \sin 2t \sec^2 t \,\mathrm{d}t$ **M1**

$= \int_0^{\frac{\pi}{4}} \frac{2 \sin t \cos t}{\cos^2 t}\,\mathrm{d}t = \int_0^{\frac{\pi}{4}} 2 \tan t \,\mathrm{d}t$ **M1**

$= \left[2 \ln |\sec t|\right]_0^{\frac{\pi}{4}} = 2 \ln \sqrt{2} - 2 \ln 1$ **M1**

$= \ln 2$ **A1***

Don't lose marks

- Integrating (bottom − top), which gives a negative answer, or subtracting only the first term of the bottom function (use brackets).
- Using the x-limits in a t-integral.
- Integrating y with respect to t without the $\frac{\mathrm{d}x}{\mathrm{d}t}$ factor.
- Using the whole area under the curve when part of the region's boundary is a tangent or other line.
- Taking the limits from where the curve meets the axis when the region is bounded by the line instead.

More tips

Memory tricks

- 'Top minus bottom' works above or below the axis, so you never need to worry about negative areas.
- Subtract the functions first, then integrate once.
- Parametric area: '$y\,\mathrm{d}x$ becomes $y\dfrac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$', and every x-limit becomes a t-limit.
- Label every point (intersections, axis crossings) on your sketch before choosing which areas to add or subtract.

Exam technique

- Many questions start 'Show that the curves meet at…': show the algebra even if you can see the answer.
- Write the integral you are going to evaluate, with its limits, before integrating: it earns the method mark.
- When the answer is given (e.g. $\ln 2$), every line must follow, including the identity you used.
- Check your area against the diagram: a rough estimate (compare with a triangle or rectangle) catches big errors.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade D

Find where two graphs intersect

Set the two equations equal and solve to get the limits of the region.

Grade C

Find the area between a curve and a line

Integrate (top − bottom), or subtract the area of a triangle or trapezium from the area under the curve.

Grade B

Find the area between two curves

Integrate the difference of the two functions between the intersection points.

Grade B

Handle regions bounded by tangents or normals

Find the line's equation and where it meets the axis, then add or subtract areas.

Grade A

Find the area under a parametric curve

Use $\int y\dfrac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$ and change the x-limits into t-limits.

Grade A*

Simplify parametric area integrals with identities

Use trigonometric identities to turn $y\dfrac{\mathrm{d}x}{\mathrm{d}t}$ into something you can integrate, e.g. $\sin 2t\sec^2 t = 2\tan t$

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