Definite integrals and areas under curves

P8.3a · Pure mathematics · revision notes

A definite integral $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ is a number, found by integrating and substituting the limits. Where the curve is above the x-axis it gives the area between the curve and the axis; where the curve is below the axis it comes out negative. Expect 4 to 6 mark questions with a sketch, often asking for an exact area.

Key facts

- $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(b) - \mathrm{F}(a)$
- Area under $y = \mathrm{f}(x)$, curve above the axis: $\int_a^b y\,\mathrm{d}x$
- Region crossing the axis: split at the roots and add the separate areas
- $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = -\int_b^a \mathrm{f}(x)\,\mathrm{d}x$
- $\ln 1 = 0$, $\ln \mathrm{e} = 1$, $\mathrm{e}^0 = 1$
- Curve below the axis: integral negative, area $= \left|\int_a^b y\,\mathrm{d}x\right|$
- $\ln a + \ln b = \ln ab$, $\ln a - \ln b = \ln \frac{a}{b}$, $k \ln a = \ln a^k$

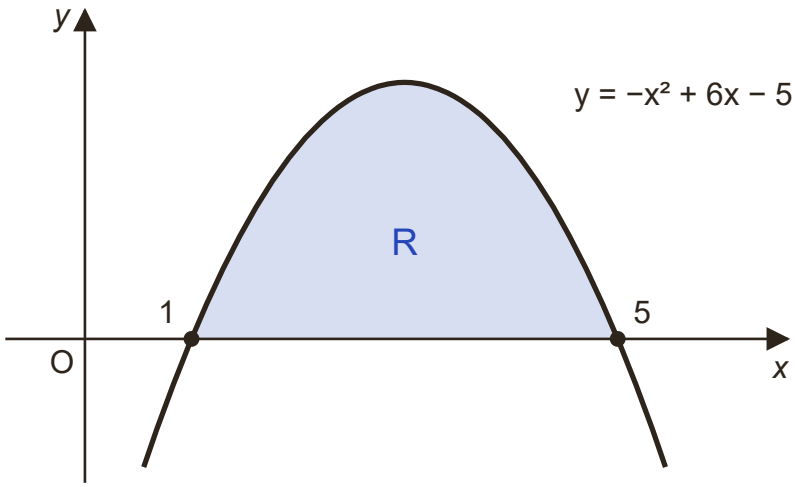
Notes

Evaluating a definite integral

- $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = [\mathrm{F}(x)]_a^b = \mathrm{F}(b) - \mathrm{F}(a)$, where F is any integral of f. There is no $+c$
- Write the integrated expression in square brackets with the limits, then substitute. Put negative limits in brackets: $(-2)^3 = -8$
- Limits for trigonometric functions are in radians. Remember $\ln 1 = 0$ and $\mathrm{e}^0 = 1$

Area under a curve

- If $y = \mathrm{f}(x) \geq 0$ for $a \leq x \leq b$, the area between the curve, the x-axis and the lines $x = a$ and $x = b$ is $\int_a^b y \, \mathrm{d}x$
- If the region is bounded by the curve and the x-axis only, the limits are the roots: solve $\mathrm{f}(x) = 0$

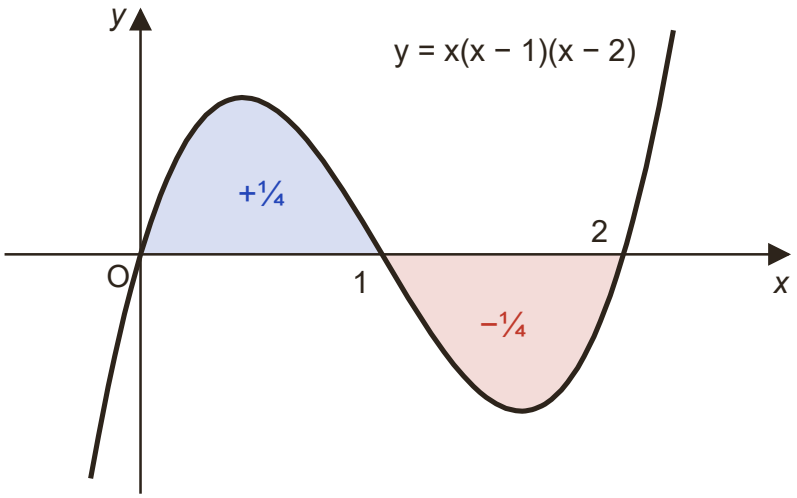


The limits are the roots: area $= \int_1^5 (-x^2 + 6x - 5) \, \mathrm{d}x = \frac{32}{3}$

- Sketch the curve first, so you can see which parts are above or below the axis.

Regions below the x-axis

- Where the curve is below the axis, $y < 0$, so $\int y \, \mathrm{d}x$ is negative. The area is the positive value: area $= \left| \int_a^b y \, \mathrm{d}x \right|$
- If a region is partly above and partly below the axis, split at the root, work out each integral separately and add the sizes of the answers.



$\int_0^1 y \, \mathrm{d}x = \frac{1}{4}$ and $\int_1^2 y \, \mathrm{d}x = -\frac{1}{4}$: the total area is $\frac{1}{2}$, but $\int_0^2 y \, \mathrm{d}x = 0$

- One integral across a root gives the net (signed) area: the parts cancel, and the result can even be zero when the total area is not.

Exact answers

- 'Exact' means leave $\ln$, e, π and surds in your answer. Tidy logs with the laws of logarithms, e.g. $3\ln 6 - 3\ln 2 = 3\ln 3 = \ln 27$

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, dM = depends on the earlier method mark, A1* = the solution must be fully correct, with every step shown.

Find the area of a region bounded by a curve and the x-axis 5 marks **Grade D**

1. Solve $y = 0$ to find the limits.
2. Integrate term by term.
3. Substitute the limits, upper minus lower.
4. Check the sign: an area is always positive.

Example. The curve C has equation $y = -x^2 + 6x - 5$. The finite region R is bounded by C and the x-axis.
Find the exact area of R

Model answer.

$-x^2 + 6x - 5 = 0 \Rightarrow -(x-1)(x-5) = 0$, so $x = 1$ and $x = 5$ **B1**

$\int_1^5 (-x^2 + 6x - 5)\,\mathrm{d}x = \left[-\frac{x^3}{3} + 3x^2 - 5x\right]_1^5$ **M1 A1**

$= \left(-\frac{125}{3} + 75 - 25\right) - \left(-\frac{1}{3} + 3 - 5\right)$

$= \frac{25}{3} - \left(-\frac{7}{3}\right)$ **dM1**

$= \frac{32}{3}$ **A1**

Find the total area of regions above and below the axis

5 marks **Grade B**

1. Find all the roots and sketch the curve to see which parts are below the axis.
2. Integrate over each part separately.
3. Add the sizes of the results.

Example. The curve C has equation $y = x(x-1)(x-2)$.
Find the total area of the two finite regions bounded by C and the x-axis.

Model answer.

$y = x^3 - 3x^2 + 2x$ **M1**

$\int y\,\mathrm{d}x = \frac{x^4}{4} - x^3 + x^2$ **M1**

Split at $x = 1$: $\int_0^1 y\,\mathrm{d}x = \frac{1}{4} - 1 + 1 = \frac{1}{4}$ and

$\int_1^2 y\,\mathrm{d}x = (4 - 8 + 4) - \frac{1}{4} = -\frac{1}{4}$ **M1 A1**

Total area $= \frac{1}{4} + \frac{1}{4}$

$= \frac{1}{2}$ **A1**

(Note that $\int_0^2 y\,\mathrm{d}x = 0$: the two parts cancel, which is why you must split.)

Show that an area has an exact value

5 marks **Grade B**

1. Rewrite the function as separate terms (split any fraction).
2. Integrate, including any $\ln$ terms.
3. Substitute the limits and simplify, using $\ln 1 = 0$, to reach the given answer.

Example. The region R is bounded by the curve with equation $y = \frac{x^2 + 4}{x}$, the x-axis and the lines $x = 1$ and $x = 3$.
Show that the area of R is $4 + 4\ln 3$

Model answer.

$y = x + 4x^{-1}$ **M1**

$\int_1^3 (x + 4x^{-1})\,\mathrm{d}x = \left[\frac{x^2}{2} + 4\ln x\right]_1^3$ **M1 A1**

$= \left(\frac{9}{2} + 4\ln 3\right) - \left(\frac{1}{2} + 4\ln 1\right)$ **dM1**

$= 4 + 4\ln 3$ **A1***

Don’t lose marks

- Integrating straight across a root and giving the net value as the area.
- Giving a negative answer for an area.
- Substituting the limits the wrong way round, or dropping the brackets around negative limits.
- Mixing up $\ln 1 = 0$ and $e^0 = 1$ when simplifying.
- Using the wrong limits: they must come from the question or from solving $y = 0$, not from the edges of the diagram.

More tips

Memory tricks

- Sketch first: the roots and the shape tell you where the curve dips below the axis.
- Use your calculator's definite integral function to check a value (it does not replace working when you are told to show all stages).
- An area can never be negative: a negative integral means the region is below the axis.
- Symmetry can halve the work: $y = \sin x$ has equal areas above and below the axis for $0 \leq x \leq 2\pi$

Exam technique

- 'Find the exact area' means no decimals: leave fractions, $\ln$ and π
- Show the square-bracket line and the two substituted values: they carry the method marks.
- If there is a diagram, check that your limits match it: roots, marked points or the lines given.
- In a context, give units: square units, or e.g. m^2.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Evaluate a definite integral of a polynomial**
Integrate, substitute the upper limit, then subtract the value at the lower limit.

Grade D — **Find the area under a curve**
For a curve above the x-axis, the area between the curve, the axis and the lines $x = a$ and $x = b$ is $\int_a^b y \, dx$

Grade C — **Find the limits where a curve meets the axis**
Solve $y = 0$ to find where a region bounded by the curve and the x-axis starts and ends.

Grade B — **Deal with regions below the x-axis**
The integral is negative there, so split the integral at each root and add the sizes of the separate answers.

Grade A — **Find exact areas involving e, ln or π**
Integrate standard functions, then simplify using laws of logarithms and exact trigonometric values.

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