Integrating standard functions

P8.2 · Pure mathematics · revision notes

You need to integrate powers of x, e^{kx}, $\frac{1}{x}$, $\sin kx$ and $\cos kx$, reverse other standard derivatives such as that of $\tan kx$, and use trigonometric identities to integrate $\sin^2 x$, $\cos^2 x$ and $\tan^2 x$. These results appear in almost every integration question on Papers 1 and 2.

Key facts

- $\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$
- $\int \frac{1}{x} \,\mathrm{d}x = \ln|x| + c$
- $\int \sin kx \,\mathrm{d}x = -\frac{1}{k}\cos kx + c$
- $\int \mathrm{f}(ax+b) \,\mathrm{d}x = \frac{1}{a}\mathrm{F}(ax+b) + c$
- $\cos^2 x = \frac{1}{2}(1+\cos 2x)$, $\sin^2 x = \frac{1}{2}(1-\cos 2x)$
- $\int \mathrm{e}^{kx} \,\mathrm{d}x = \frac{1}{k}\mathrm{e}^{kx} + c$
- $\int \cos kx \,\mathrm{d}x = \frac{1}{k}\sin kx + c$
- $\int \sec^2 kx \,\mathrm{d}x = \frac{1}{k}\tan kx + c$
- $\tan^2 x = \sec^2 x - 1$

Notes

Powers of x

- $\int x^n \,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$ works for every n except -1, including fractional and negative powers.
- Rewrite first: $\sqrt{x} = x^{\frac{1}{2}}$, $\frac{3}{x^2} = 3x^{-2}$, $\frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$. Expand brackets and split fractions that have a single term underneath.

Exponentials and 1/x

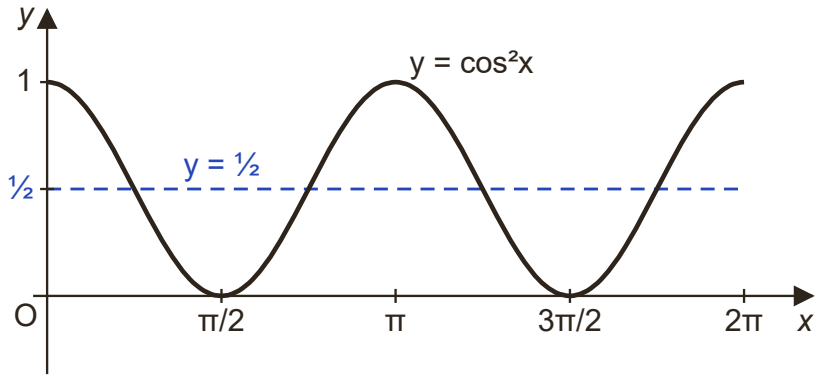
- $\int \mathrm{e}^{kx} \,\mathrm{d}x = \frac{1}{k}\mathrm{e}^{kx} + c$: divide by k, because differentiating multiplies by k
- The missing power, $n = -1$: $\int \frac{1}{x} \,\mathrm{d}x = \ln|x| + c$. The modulus is there because $\ln$ is only defined for positive numbers; if $x > 0$ you can write $\ln x$
- $\int \frac{k}{x} \,\mathrm{d}x = k\ln|x| + c$ and $\int \frac{1}{ax+b} \,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$

Trigonometric functions (radians only)

- $\int \cos kx \, \mathrm{d}x = \frac{1}{k}\sin kx + c$ and $\int \sin kx \, \mathrm{d}x = -\frac{1}{k}\cos kx + c$
- Reverse the derivatives you know: $\int \sec^2 kx \, \mathrm{d}x = \frac{1}{k}\tan kx + c$, $\int \sec x \tan x \, \mathrm{d}x = \sec x + c$, $\int \operatorname{cosec}^2 x \, \mathrm{d}x = -\cot x + c$, $\int \operatorname{cosec} x \cot x \, \mathrm{d}x = -\operatorname{cosec} x + c$
- Some results, such as $\int \tan kx \, \mathrm{d}x = \frac{1}{k}\ln|\sec kx| + c$, are in the formula booklet: know where to find them.

Using identities first

- $\sin^2 x$, $\cos^2 x$ and $\tan^2 x$ cannot be integrated directly. Rewrite them: $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\tan^2 x = \sec^2 x - 1$



$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$: a $\cos 2x$ wave of height $\frac{1}{2}$ about $y = \frac{1}{2}$, which is easy to integrate.

- So $\int \sin^2 x \, \mathrm{d}x = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$ and $\int \tan^2 x \, \mathrm{d}x = \tan x - x + c$
- Also useful: $\sin x \cos x = \frac{1}{2}\sin 2x$

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, dM = depends on the earlier method mark, A1* = the solution must be fully correct, with every step shown.

Integrate a sum of standard functions

3 to 4 marks | **Grade D**

1. Split the integrand into separate terms.
2. Integrate each term with its standard result, dividing by the coefficient of x inside.
3. Add a single $+c$ at the end.

Example. Find $\int \left(4\mathrm{e}^{2x} - \frac{6}{x} + 3\cos 3x\right)\mathrm{d}x,\ x > 0$

Model answer.

$2\mathrm{e}^{2x}$	**B1**
$-6\ln x$	**B1**
$3\cos 3x$ integrates to the form $k\sin 3x$	**M1**
$2\mathrm{e}^{2x} - 6\ln x + \sin 3x + c$	**A1**

Show that a definite integral has an exact value

4 to 5 marks | **Grade C**

1. Integrate each term, including the $\frac{1}{a}$ from any linear expression inside.
2. Substitute the upper limit, then subtract the value at the lower limit.
3. Use $\mathrm{e}^0 = 1$ and $\ln 1 = 0$, and simplify to the given form, showing each step.

Example. Show that $\int_0^1 \left(\mathrm{e}^{2x} + \frac{4}{2x+1}\right)\mathrm{d}x = \frac{1}{2}\left(\mathrm{e}^2 - 1\right) + 2\ln 3$

Model answer.

$\left[\frac{1}{2}\mathrm{e}^{2x} + 2\ln(2x+1)\right]_0^1$ **B1** (for $\frac{1}{2}\mathrm{e}^{2x}$) **M1** (for $k\ln(2x+1)$) **A1** (for $2\ln(2x+1)$)

$= \left(\frac{1}{2}\mathrm{e}^2 + 2\ln 3\right) - \left(\frac{1}{2} + 2\ln 1\right)$ **dM1**

$= \frac{1}{2}\left(\mathrm{e}^2 - 1\right) + 2\ln 3$ **A1***

Use a trigonometric identity to integrate 5 marks **Grade A**

1. Spot $\sin^2$, $\cos^2$ or $\tan^2$: these need an identity first.
2. Rewrite using a double angle formula or $\tan^2 x = \sec^2 x - 1$ and simplify.
3. Integrate, then use exact values of $\sin$, $\cos$ and $\tan$ at the limits (in radians).

Example. Find the exact value of $\int_0^{\frac{\pi}{4}} \left(2\cos^2 x + \tan^2 x\right) \mathrm{d}x$

Model answer.

$2\cos^2 x = 1 + \cos 2x$ **B1** and $\tan^2 x = \sec^2 x - 1$ **B1**

So the integrand is $\cos 2x + \sec^2 x$

$$\left[\frac{1}{2}\sin 2x + \tan x\right]_0^{\frac{\pi}{4}}$$ **M1 A1**

$$= \left(\frac{1}{2} + 1\right) - 0$$

$$= \frac{3}{2}$$ **A1**

Don't lose marks

- Multiplying by k instead of dividing: $\int \mathrm{e}^{3x}\,\mathrm{d}x = \frac{1}{3}\mathrm{e}^{3x} + c$, not $3\mathrm{e}^{3x} + c$
- Getting the sign wrong: $\int \sin x\,\mathrm{d}x = -\cos x + c$
- Using the power rule on $\frac{1}{x}$, which would divide by zero; the answer is $\ln|x|$. But $\frac{1}{x^2} = x^{-2}$ does use the power rule: $-x^{-1} + c$
- Treating $\sin^2 x$ like a power of x: $\int \sin^2 x\,\mathrm{d}x$ is not $\frac{1}{3}\sin^3 x + c$
- Using the 'divide by a' rule when the inside is not linear: $\int \mathrm{e}^{x^2}\,\mathrm{d}x$ is not $\frac{1}{2x}\mathrm{e}^{x^2} + c$
- Working in degrees: the trigonometric integrals only hold in radians.

More tips

Memory tricks

- Linear expression inside? Integrate as normal, then divide by the coefficient of x
- Differentiating goes $\sin \to \cos \to -\sin \to -\cos$; integrating goes the other way, so only the integral of $\sin$ picks up a minus sign.
- Differentiate your answer to check: $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}\mathrm{e}^{3x}\right) = \mathrm{e}^{3x}$ confirms the $\frac{1}{3}$
- A squared trigonometric function in an integral? Reach for an identity straight away.

Exam technique

- Write $\ln|x|$, or $\ln x$ when you know $x > 0$. Leave answers exact ($\ln$, e, π, surds) unless a decimal is asked for.
- An integral with no limits needs $+c$ in the final answer.
- In a 'show that', write the square-bracket line and the substituted values: the final line alone earns nothing.
- Know the basic results by heart, and check before the exam which extra results the formula booklet gives you.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Integrate powers of x, including fractional powers**

Rewrite roots and fractions as powers first, e.g. $\frac{2}{\sqrt{x}} = 2x^{-\frac{1}{2}}$, which integrates to $4x^{\frac{1}{2}} + c$

Grade D — **Integrate exponentials and 1/x**

$\int \mathrm{e}^{kx}\,\mathrm{d}x = \frac{1}{k}\mathrm{e}^{kx} + c$ and $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c$

Grade C — **Integrate sin kx and cos kx in radians**

$\int \cos kx\,\mathrm{d}x = \frac{1}{k}\sin kx + c$ and $\int \sin kx\,\mathrm{d}x = -\frac{1}{k}\cos kx + c$

Grade C — **Integrate functions of a linear expression**

If $\int \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(x) + c$, then $\int \mathrm{f}(ax+b)\,\mathrm{d}x = \frac{1}{a}\mathrm{F}(ax+b) + c$, e.g.

$\int \frac{1}{3x-2}\,\mathrm{d}x = \frac{1}{3}\ln|3x-2| + c$

Grade B — **Reverse standard derivatives such as tan kx**

Because $\frac{\mathrm{d}}{\mathrm{d}x}(\tan kx) = k\sec^2 kx$, $\int \sec^2 kx\,\mathrm{d}x = \frac{1}{k}\tan kx + c$

Grade A — **Use trigonometric identities before integrating**

Replace $\sin^2 x$, $\cos^2 x$ or $\tan^2 x$ using the double angle formulae or $\tan^2 x = \sec^2 x - 1$, then integrate.

Free notes, Full packs, practice papers and mock exams at chhetriacademy.com. Written for this site; not taken from past papers. Stuck? 1-to-1 tutoring with a free 30-minute trial lesson.