The fundamental theorem of calculus

P8.1 · Pure mathematics · revision notes

Integration is the reverse of differentiation, and the fundamental theorem of calculus links the two: if $\mathrm{F}'(x) = \mathrm{f}(x)$ then $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(b) - \mathrm{F}(a)$. On Papers 1 and 2 it usually appears as finding a curve from its gradient function, or as the first step of a longer definite integral question.

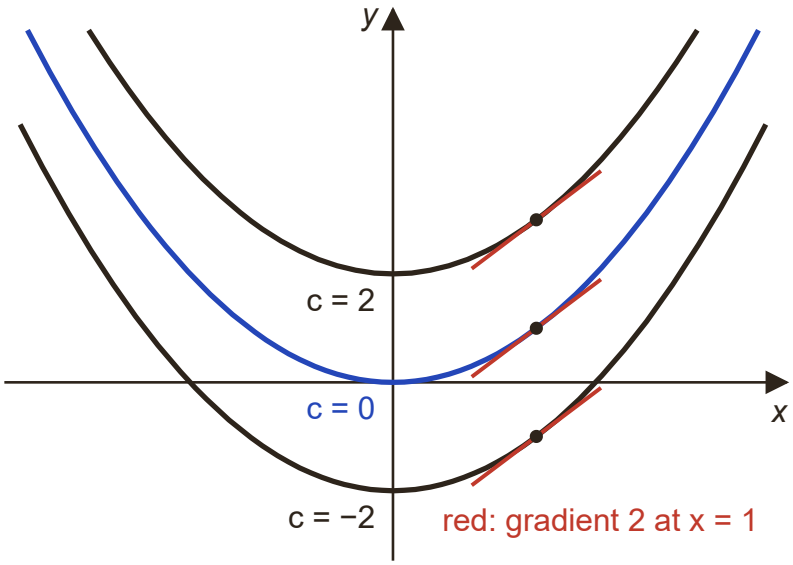
Key facts

- $\int \mathrm{f}'(x)\,\mathrm{d}x = \mathrm{f}(x) + c$
- $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(b) - \mathrm{F}(a)$, where $\mathrm{F}'(x) = \mathrm{f}(x)$
- $\int_a^b \mathrm{f}(x)\,\mathrm{d}x + \int_b^c \mathrm{f}(x)\,\mathrm{d}x = \int_a^c \mathrm{f}(x)\,\mathrm{d}x$
- $\int k\,\mathrm{f}(x)\,\mathrm{d}x = k\int \mathrm{f}(x)\,\mathrm{d}x$; integrate sums term by term
- Stationary point at $x = p$: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = p$
- $\int x^n\,\mathrm{d}x = \dfrac{x^{n+1}}{n+1} + c,\ n \neq -1$
- $\int_b^a \mathrm{f}(x)\,\mathrm{d}x = -\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ and $\int_a^a \mathrm{f}(x)\,\mathrm{d}x = 0$
- Roots and fractions as powers: $\sqrt[3]{x^2} = x^{\frac{2}{3}}$, $\dfrac{5}{x^3} = 5x^{-3}$

Notes

Integration reverses differentiation

- If $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x)$, then $y = \mathrm{f}(x) + c$. In symbols, $\int \mathrm{f}'(x)\,\mathrm{d}x = \mathrm{f}(x) + c$
- $y = x^2 + 1$, $y = x^2 - 7$ and $y = x^2$ all have gradient $2x$. Integrating $2x$ gives the whole family $y = x^2 + c$, one curve for each value of the **constant of integration** c

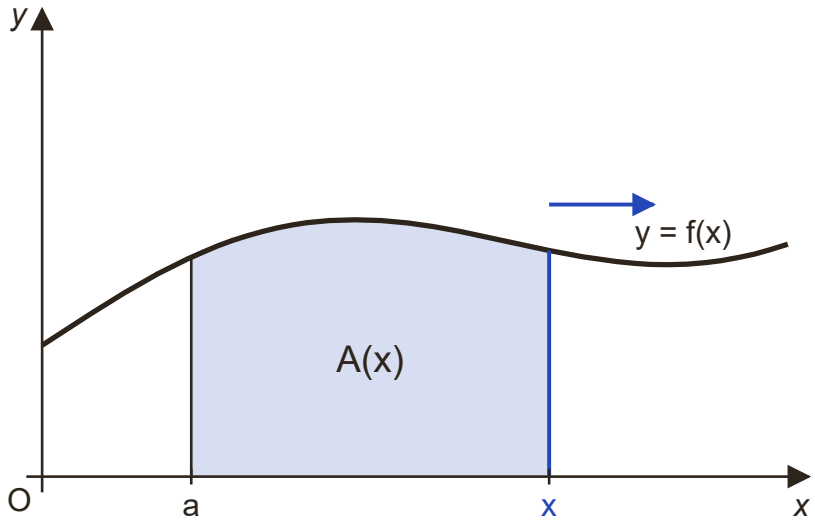


Every curve $y = x^2 + c$ has gradient $2x$: at any given x their tangents are parallel.

- Powers: $\int x^n\,\mathrm{d}x = \dfrac{x^{n+1}}{n+1} + c$ for $n \neq -1$. Raise the power by 1, then divide by the new power.
- You can integrate a sum term by term and take constant factors outside, but not a product or quotient factor by factor. Expand brackets and split fractions into powers first, e.g. $\dfrac{x^2+3}{\sqrt{x}} = x^{\frac{3}{2}} + 3x^{-\frac{1}{2}}$
- Check any answer by differentiating it: you should get back exactly what you integrated.

The fundamental theorem of calculus

- If $\mathbf{F}'(x) = \mathbf{f}(x)$, then $\int_a^b \mathbf{f}(x)\,\mathrm{d}x = [\mathbf{F}(x)]_a^b = \mathbf{F}(b) - \mathbf{F}(a)$
- The constant cancels, $(\mathbf{F}(b) + c) - (\mathbf{F}(a) + c) = \mathbf{F}(b) - \mathbf{F}(a)$, so leave out c in definite integrals.
- An indefinite integral is a family of functions; a definite integral is a number.
- Why it works: the area under $y = \mathbf{f}(x)$ from a up to a moving value x grows at the rate $\mathbf{f}(x)$, so this area function is an integral of $\mathbf{f}$. That is why integrating finds areas.



As x moves right, the shaded area $A(x)$ grows at the rate $f(x)$, so $A'(x) = f(x)$

- Swapping the limits changes the sign: $\int_b^a \mathbf{f}(x)\,\mathrm{d}x = -\int_a^b \mathbf{f}(x)\,\mathrm{d}x$

Finding a particular curve

- Given $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and a point on the curve: integrate, substitute the point, solve for c, then write the full equation $y = \ldots$
- Given $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$: integrate once to get $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ (constant A), then again to get y (constant B). A stationary point at $x = p$ tells you $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = p$
- Find each constant as soon as you can: it keeps the algebra short.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Find the equation of a curve from its gradient function

4 to 5 marks | Grade D

1. Write every term as a power of x
2. Integrate term by term and add c
3. Substitute the given point and solve for c
4. Write the final answer as $y = \ldots$ (or $\mathrm{f}(x) = \ldots$) with the value of c in it.

Example. The curve C has equation $y = \mathrm{f}(x)$, $x > 0$, and passes through the point $(4, 9)$.
Given that $\mathrm{f}'(x) = 3\sqrt{x} - \dfrac{8}{x^2}$, find $\mathrm{f}(x)$

Model answer.

$\mathrm{f}'(x) = 3x^{\frac{1}{2}} - 8x^{-2}$

$\mathrm{f}(x) = \dfrac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{8x^{-1}}{-1} + c = 2x^{\frac{3}{2}} + 8x^{-1} + c$ **M1** (a power raised by 1) **A1** (one term correct) **A1** (both terms correct)

At $(4, 9)$: $2(8) + \dfrac{8}{4} + c = 9$, so $c = -9$ **M1**

$\mathrm{f}(x) = 2x^{\frac{3}{2}} + \dfrac{8}{x} - 9$ **A1**

Integrate twice from a second derivative

5 to 6 marks | Grade B

1. Integrate $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ to get $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, with a constant A
2. Use the gradient information (e.g. a stationary point) to find A
3. Integrate again, with a new constant B, and use the point on the curve to find B

Example. A curve has $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x - 4$. The point $(1, 5)$ is a stationary point of the curve.
Find an equation of the curve.

Model answer.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 4x + A$ **M1 A1**

Stationary at $x = 1$: $3 - 4 + A = 0$, so $A = 1$ **B1**

$y = x^3 - 2x^2 + x + B$; at $(1, 5)$: $1 - 2 + 1 + B = 5$ **M1**

$y = x^3 - 2x^2 + x + 5$ **A1**

Use a definite integral to find an unknown 4 marks **Grade A**

1. Integrate and write the result in square brackets with the limits.
2. Substitute the limits (upper minus lower) and set the result equal to the given value.
3. Solve, and reject any value that breaks a condition in the question, saying why.

Example. Given that $k > 1$ and $\int_1^k (4x - 3)\,\mathrm{d}x = 15$, find the value of k

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Model answer.

$\left[2x^2 - 3x\right]_1^k$ **M1 A1**

$(2k^2 - 3k) - (2 - 3) = 15 \Rightarrow 2k^2 - 3k - 14 = 0$ **M1**

$(2k - 7)(k + 2) = 0$, so $k = \dfrac{7}{2}$ (reject $k = -2$ because $k > 1$) **A1**

Don't lose marks

- Leaving out $+ c$ in an indefinite integral, or never finding it when the question gives a point on the curve.
- Integrating a product factor by factor: $\int (x + 1)(x - 2)\,\mathrm{d}x$ is not $\int (x + 1)\,\mathrm{d}x \times \int (x - 2)\,\mathrm{d}x$. Expand first.
- Slips with negative powers: $\int x^{-3}\,\mathrm{d}x = \dfrac{x^{-2}}{-2} + c$, not $\dfrac{x^{-4}}{-4} + c$
- Subtracting the wrong way round, $\mathrm{F}(a) - \mathrm{F}(b)$, which flips the sign of the answer.
- Stopping once you have found c when the question asks for the equation of the curve: the answer is the full equation, e.g. $y = 2x^{\frac{3}{2}} + \dfrac{8}{x} - 9$

More tips

Memory tricks

- Powers: 'add one to the power, divide by the new power'.
- Differentiate your answer to check it: it takes seconds and catches most slips.
- Definite integrals are 'top minus bottom': $\mathbf{F(upper) - F(lower)}$. Put negative limits in brackets.
- 'Passes through the point' is your cue to find c. 'Stationary point' gives you $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
- Your calculator can evaluate a definite integral: use it to check an exact answer you found by hand.

Exam technique

- 'Find y in terms of x' or 'find $\mathbf{f}(x)$' needs the complete function, including the value of the constant.
- Write down the integrated expression before substituting anything: that line earns the method marks even if the arithmetic later goes wrong.
- When a question says 'solutions relying entirely on calculator technology are not acceptable', show the integration and the substitution of the limits in full; use the calculator only to check.
- Simplify coefficients such as $\frac{3}{3/2} = 2$ before substituting a point: it makes the arithmetic easier.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Integrate as the reverse of differentiation**

If $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2$ then $y = 2x^3 + c$, because differentiating $2x^3 + c$ gives back $6x^2$

Grade E — **Add a constant to every indefinite integral**

The derivative of any constant is zero, so an indefinite integral is only known up to $+ c$

Grade D — **Find a curve through a given point**

Integrate the gradient function, then substitute the coordinates of the point to find c

Grade C — **Evaluate a definite integral as F(b) − F(a)**

Find an integral $\mathbf{F}$, substitute the upper limit and subtract its value at the lower limit; no constant is needed.

Grade B — **Integrate twice from a second derivative**

Each integration brings a new constant, so you need two conditions, such as a point on the curve and a stationary point.

Grade A — **Find an unknown limit or constant**

Set a definite integral equal to its given value, solve the equation and reject any value that breaks a given condition.

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