Constructing differential equations

P7.6 · Pure mathematics · revision notes

Turning a description in words into a differential equation, in pure maths and in contexts such as population growth, cooling, draining tanks, kinematics and price and demand. In the specification this is point 7.6. You only need to form (and sometimes interpret) the equation here; solving it is covered in Integration.

Key facts

- Rate of change of x with respect to t: $\frac{\mathrm{d}x}{\mathrm{d}t}$
- Rate proportional to x: $\frac{\mathrm{d}x}{\mathrm{d}t} = kx$; inversely proportional: $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{k}{x}$
- Decreasing: $\frac{\mathrm{d}x}{\mathrm{d}t} = -kx$ with $k > 0$
- Newton's law of cooling: $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - \theta_0)$
- Net rate = rate in − rate out
- $v = \frac{\mathrm{d}s}{\mathrm{d}t}$, $a = \frac{\mathrm{d}v}{\mathrm{d}t}$
- Change of variable: $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}h}$

Notes

From words to a differential equation

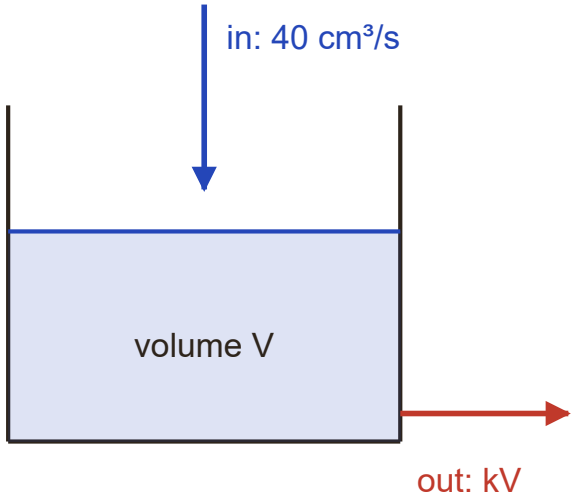
- A differential equation links a quantity and its rate of change. 'The rate of change of x' with respect to time is $\frac{\mathrm{d}x}{\mathrm{d}t}$
- 'Proportional to' means 'equals a constant times': 'the rate of increase of P is proportional to P' gives $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$, where $k > 0$
- 'Inversely proportional to x' gives $\frac{k}{x}$; 'proportional to the square of x' gives kx^2
- If the quantity is decreasing, its derivative is negative: write $\frac{\mathrm{d}m}{\mathrm{d}t} = -km$ and state that k is a positive constant.
- Newton's law of cooling: the rate of decrease of the temperature θ is proportional to the difference between θ and the room temperature θ_0, so $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - \theta_0)$

Common contexts

- Kinematics: $v = \dfrac{\mathrm{d}s}{\mathrm{d}t}$ and $a = \dfrac{\mathrm{d}v}{\mathrm{d}t}$. A particle slowed only by a resistance proportional to its speed gives $\dfrac{\mathrm{d}v}{\mathrm{d}t} = -kv$
- Population: rate of change = births − deaths, e.g. births proportional to P and deaths proportional to P^2 give $\dfrac{\mathrm{d}P}{\mathrm{d}t} = aP - bP^2$
- Price and demand: if demand D falls as the price p rises, at a rate proportional to D, then $\dfrac{\mathrm{d}D}{\mathrm{d}p} = -kD$
- Pure maths: 'the gradient at every point is proportional to xy' gives $\dfrac{\mathrm{d}y}{\mathrm{d}x} = kxy$

Combining rates and finding constants

- Net rate = rate in − rate out: liquid poured in at 40 cm^3 s^{-1} and leaking at a rate proportional to V gives $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 40 - kV$



Net rate = rate in − rate out: $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 40 - kV$

- If the rate is given for one variable but the question is about another, use the chain rule, e.g. $\dfrac{\mathrm{d}h}{\mathrm{d}t} = \dfrac{\mathrm{d}V}{\mathrm{d}t} \div \dfrac{\mathrm{d}V}{\mathrm{d}h}$
- Find a constant by substituting a known rate and the value of the variable at the same moment.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Write down a differential equation and find the constant

3 to 4 marks | **Grade C**

1. Write the rate as a derivative, e.g. $\dfrac{\mathrm{d}m}{\mathrm{d}t}$
2. Replace 'proportional to' with 'a constant times'.
3. Put in a minus sign for a decrease and state that $k > 0$
4. Substitute the given rate and value at the same instant to find k

Example. A block of ice is melting. At time t minutes its mass is m grams. The mass decreases at a rate that is proportional to the square root of the mass.
(a) Write down a differential equation connecting m and t.
(b) When $m = 400$, the mass is decreasing at 5 grams per minute. Find the value of the constant of proportionality.

Model answer.

(a) $\dfrac{\mathrm{d}m}{\mathrm{d}t} = -k\sqrt{m}$, where k is a positive constant B1 (derivative and $\sqrt{m}$) B1 (negative sign with $k > 0$)

(b) $-5 = -k\sqrt{400}$

$= -20k$ M1, so $k = 0.25$ A1

Inflow and outflow

4 to 5 marks | **Grade A**

1. Write net rate = rate in − rate out, with a constant for the proportional part.
2. Use the given moment to find the constant.
3. Rearrange to the given form.
4. For long-term behaviour, find where the rate is 0 and think about the sign of the rate on each side.

Example. Liquid is poured into a large tank at a constant rate of 40 cm^3 s^{-1}. At the same time liquid leaks out at a rate proportional to the volume, V cm^3, of liquid in the tank. When $V = 1000$, the volume is increasing at 15 cm^3 s^{-1}.

(a) Show that $\dfrac{\mathrm{d}V}{\mathrm{d}t} = \dfrac{1600 - V}{40}$.

(b) Explain what the differential equation suggests about the volume in the long term.

Model answer.

(a) $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 40 - kV$ B1

$15 = 40 - 1000k$ M1, so $k = \dfrac{1}{40}$ A1

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 40 - \frac{V}{40} = \frac{1600 - V}{40}$$ A1

(b) While $V < 1600$ the rate is positive but gets smaller as V gets closer to 1600, so the volume increases towards 1600 cm^3 B1

Use the chain rule to form an equation

5 to 6 marks **Grade A**

1. Write the volume in terms of the depth and differentiate.
2. Write the given rate using its own constant, with a minus sign for a decrease.
3. Use the chain rule to get $\frac{\mathrm{d}h}{\mathrm{d}t}$, then combine the constants into one.
4. Substitute the given values to find the constant.

Example. Water leaks from a hole in the base of a cylindrical tank of radius 20 cm. The depth of water is h cm at time t minutes. The volume of water decreases at a rate proportional to $\sqrt{h}$.

(a) Show that $\frac{\mathrm{d}h}{\mathrm{d}t} = -k\sqrt{h}$, where k is a positive constant.

(b) When $h = 25$, the depth is falling at 0.3 cm per minute. Find the value of k

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Model answer.

(a) $V = 400\pi h$, so $\frac{\mathrm{d}V}{\mathrm{d}h} = 400\pi$ B1

$\frac{\mathrm{d}V}{\mathrm{d}t} = -c\sqrt{h}$ for a positive constant c B1

$\frac{\mathrm{d}h}{\mathrm{d}t}$

$= \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}h}$

$= -\frac{c}{400\pi}\sqrt{h}$ M1

$= -k\sqrt{h}$, where $k = \frac{c}{400\pi}$ is a positive constant A1

(b) $-0.3 = -k\sqrt{25}$ M1, so $k = 0.06$ A1

Gradient conditions in pure maths

3 marks | **Grade B**

1. 'Gradient' means $\dfrac{\mathrm{d}y}{\mathrm{d}x}$
2. Translate the description into an expression with a constant k
3. Substitute the point and its gradient to find k

Example. At every point (x, y) on a curve, the gradient of the curve is proportional to the product of x and y. The point $(2, 3)$ lies on the curve, and the gradient of the curve at this point is 4. Find a differential equation satisfied by the curve.

Model answer.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = kxy$ B1

$4 = k \times 2 \times 3$ M1, so $k = \dfrac{2}{3}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2}{3}xy$ A1

Don't lose marks

- Leaving out the minus sign for a decrease, or using both a minus sign and a negative k
- Writing the quantity instead of its rate, e.g. $m = -k\sqrt{m}$ instead of $\dfrac{\mathrm{d}m}{\mathrm{d}t} = -k\sqrt{m}$
- Writing $\dfrac{\mathrm{d}h}{\mathrm{d}t}$ when the given rate is for the volume, without using the chain rule.
- Finding k from values at different moments, or substituting a value where a rate is needed.
- Not saying what the constant is, e.g. 'where k is a positive constant'.

More tips

Memory tricks

- Translate word by word: 'rate of change of V' becomes $\dfrac{\mathrm{d}V}{\mathrm{d}t}$; 'is proportional to' becomes '$= k\times$'.
- Decreasing? Put the minus sign in front and keep k positive.
- Check the units: both sides of the equation must have the same units, e.g. $\mathrm{cm}^3\ \mathrm{s}^{-1}$.
- Setting the rate equal to 0 tells you the value at which the quantity stops changing.

Exam technique

- Define every constant you introduce and state its sign.
- In 'show that' questions, explain where each term comes from (e.g. 'rate in' and 'rate out').
- Read carefully: 'with respect to time' and 'with respect to price' need different derivatives.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade C **Translate 'rate of change' into derivative notation**

'The rate of change of P with respect to time' is $\frac{\mathrm{d}P}{\mathrm{d}t}$

Grade C **Write 'proportional to' using a constant**

'Proportional to P' becomes kP; 'inversely proportional to $\sqrt{h}$' becomes $\frac{k}{\sqrt{h}}$

Grade C **Use a negative sign for a decrease**

A decreasing quantity gives, for example, $\frac{\mathrm{d}m}{\mathrm{d}t} = -km$, where $k > 0$

Grade B **Find the constant of proportionality**

Substitute a known rate and value at the same moment, e.g. $-5 = -k\sqrt{400}$ gives $k = 0.25$

Grade A **Form equations with inflow and outflow**

Net rate = rate in − rate out, e.g. $\frac{\mathrm{d}V}{\mathrm{d}t} = 40 - kV$

Grade A **Use the chain rule to change the variable**

Turn a rate of change of volume into a rate of change of depth with $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}h}$

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