Connected rates of change

P7.4b · Pure mathematics · revision notes

Connected rates of change use the chain rule to link two rates, for example how fast the radius of a balloon grows when you know how fast its volume grows. In the specification this is part of point 7.4. Questions are usually 4 to 6 marks and often involve spheres, cones, cylinders or cubes.

Key facts

- $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$
- $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\mathrm{d}y/\mathrm{d}x}$
- Sphere: $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$
- Cone: $V = \frac{1}{3}\pi r^2 h$
- Cylinder: $V = \pi r^2 h$
- Circle: $A = \pi r^2$, $C = 2\pi r$
- A decreasing quantity has a negative rate of change

Notes

Rates of change

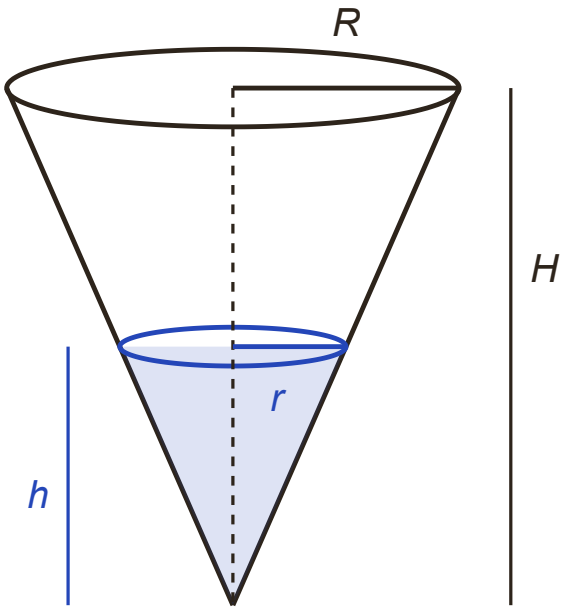
- A rate of change with respect to time is a derivative with respect to t: $\frac{\mathrm{d}V}{\mathrm{d}t}$ is the rate of change of volume.
- ‘Increasing at 3 cm^3 s^{-1}’ means $\frac{\mathrm{d}V}{\mathrm{d}t} = 3$. ‘Decreasing at 2 cm s^{-1}’ means $\frac{\mathrm{d}r}{\mathrm{d}t} = -2$

Linking rates with the chain rule

- Connect the rate you want to the rate you know through a variable they share: $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$
- The middle derivative comes from differentiating a formula, e.g. $V = \frac{4}{3}\pi r^3$ gives $\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$
- Turn a derivative upside down when you need to: $\frac{\mathrm{d}r}{\mathrm{d}V} = \frac{1}{\mathrm{d}V/\mathrm{d}r}$
- Chains can be longer: $\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}S}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$

Method

- Write down the rate you know and the rate you want, in derivative notation.
- Find a formula linking the two variables. If there is a third variable, eliminate it first (e.g. by similar triangles in a cone).



Similar triangles: $\frac{r}{h} = \frac{R}{H}$, so r can be written in terms of h and eliminated.

- Differentiate the formula, write the chain rule, then substitute the value at the given instant. Substitute only after differentiating.
- Give units: for example $\text{cm}^3\ \text{s}^{-1} \div \text{cm}^2$ gives $\text{cm}\ \text{s}^{-1}$.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Find a rate at an instant

4 marks | Grade B

1. Write the given rate and the required rate in derivative notation.
2. Differentiate the formula linking the variables.
3. Write the chain rule so the derivatives link up, then substitute.

Example. A spherical balloon is inflated so that its volume increases at a constant rate of 50 $\text{cm}^3\ \text{s}^{-1}$. Find the rate at which the radius is increasing at the instant when the radius is 10 cm. [The volume of a sphere is $V = \frac{4}{3}\pi r^3$.]

Model answer.

$\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$ B1

$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \div \frac{\mathrm{d}V}{\mathrm{d}r}$

$= \frac{50}{4\pi r^2}$ M1

At $r = 10$: $\frac{50}{400\pi}$ M1 $= \frac{1}{8\pi} = 0.0398$ cm s^{-1} (3 s.f.) A1

Rate of change of area from a rate of change of length

3 marks | Grade C

1. Differentiate the area formula with respect to r
2. Multiply by $\frac{\mathrm{d}r}{\mathrm{d}t}$
3. Substitute the radius and give units.

Example. The radius of a circular ink blot increases at a rate of 0.2 cm s^{-1}. Find the rate at which its area is increasing when the radius is 3 cm.

Model answer.

$A = \pi r^2 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$ B1

$\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r \times 0.2$ M1

At $r = 3$: $1.2\pi = 3.77$ $\text{cm}^2\ \text{s}^{-1}$ (3 s.f.) A1

Water in a cone: similar triangles

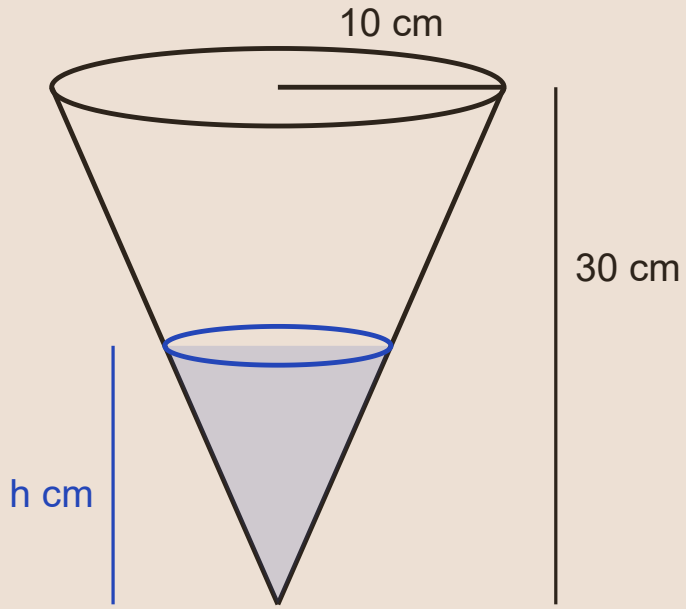
5 to 6 marks | **Grade A**

1. Use similar triangles to write the radius in terms of the depth.
2. Substitute into the volume formula so that V depends on h only.
3. Differentiate with respect to h, then use $\dfrac{\mathrm{d}h}{\mathrm{d}t} = \dfrac{\mathrm{d}V}{\mathrm{d}t} \div \dfrac{\mathrm{d}V}{\mathrm{d}h}$
4. Substitute the depth last.

Example. Water is poured at a constant rate of 12 cm^3 s^{-1} into an empty container shaped like an inverted right circular cone. The cone has height 30 cm and the radius of its top is 10 cm. When the depth of water is h cm, the volume of water is V cm^3.

(a) Show that $V = \dfrac{\pi h^3}{27}$.

(b) Find the exact rate at which the depth is increasing when $h = 6$.



Model answer.

(a) By similar triangles, the radius of the water surface is $r = \dfrac{10}{30}h = \dfrac{h}{3}$ M1

$$V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h$$

$$= \frac{\pi h^3}{27}$$ A1

(b) $\dfrac{\mathrm{d}V}{\mathrm{d}h} = \dfrac{\pi h^2}{9}$ B1

$$\frac{\mathrm{d}h}{\mathrm{d}t}$$
$$= 12 \div \frac{\pi h^2}{9}$$
$$= \frac{108}{\pi h^2}$$ M1

At $h = 6$: $\frac{108}{36\pi} = \frac{3}{\pi}$ cm s^{-1} A1

A longer chain of rates 4 to 5 marks **Grade A**

1. Choose a variable that links both quantities (here the edge length x).
2. Write both formulas in terms of x and differentiate each.
3. Find $\frac{\mathrm{d}x}{\mathrm{d}t}$ from the known rate, then the required rate.
4. Use a negative rate for a decrease and interpret the sign at the end.

Example. The surface area of a cube is decreasing at a constant rate of 6 cm^2 s^{-1}. Find the rate of change of the volume of the cube at the instant when its edges are 5 cm long.

Model answer.

Edge x cm: $A = 6x^2$ and $V = x^3$, so $\frac{\mathrm{d}A}{\mathrm{d}x} = 12x$ and $\frac{\mathrm{d}V}{\mathrm{d}x} = 3x^2$ B1

$$\frac{\mathrm{d}x}{\mathrm{d}t}$$
$$= \frac{\mathrm{d}A}{\mathrm{d}t} \div \frac{\mathrm{d}A}{\mathrm{d}x}$$
$$= \frac{-6}{12x}$$
$$= -\frac{1}{2x}$$ M1

$$\frac{\mathrm{d}V}{\mathrm{d}t}$$
$$= 3x^2 \times \left(-\frac{1}{2x}\right)$$
$$= -\frac{3x}{2}$$ M1

At $x = 5$: $\frac{\mathrm{d}V}{\mathrm{d}t} = -7.5$, so the volume is decreasing at 7.5 cm^3 s^{-1} A1

Don’t lose marks

- Substituting the value (e.g. $h = 6$) before differentiating, which makes the volume a constant.

- Using a positive rate for something that is decreasing.
- Leaving a third variable in the formula instead of eliminating it first.
- Multiplying by $\frac{\mathrm{d}V}{\mathrm{d}r}$ when you should divide by it.
- Missing or wrong units in the final answer.

More tips

Memory tricks

- Write the chain so that the derivatives 'cancel' like fractions: $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$
- Check the units come out right, e.g. $\mathrm{cm}^3\ \mathrm{s}^{-1} \div \mathrm{cm}^2 = \mathrm{cm}\ \mathrm{s}^{-1}$.
- Differentiate first, substitute last.

Exam technique

- Write each derivative with its name (e.g. $\frac{\mathrm{d}V}{\mathrm{d}h} = \ldots$) so the examiner can award method marks.
- Give exact answers such as $\frac{3}{\pi}$ when asked; otherwise give 3 significant figures, with units.
- In a 'show that' for a formula, explain the similar triangles step in words.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade C — **Write a rate using derivative notation**

'Volume increasing at 3 $\mathrm{cm}^3\ \mathrm{s}^{-1}$' means $\frac{\mathrm{d}V}{\mathrm{d}t} = 3$; a decrease gives a negative derivative.

Grade C — **Connect rates with the chain rule**

E.g. $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$

Grade B — **Find an unknown rate at an instant**

Differentiate the formula first, then substitute the value at that instant.

Grade A — **Use similar triangles to link variables**

In a cone of water, write r in terms of h before finding $\frac{\mathrm{d}V}{\mathrm{d}h}$

Grade A — **Use chains of three or more rates**

E.g. $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}A} \times \frac{\mathrm{d}A}{\mathrm{d}t}$

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