Implicit and parametric differentiation

P7.5 · Pure mathematics · revision notes

Differentiating curves given by parametric equations (x and y both in terms of t) and curves given implicitly (an equation mixing x and y). Only first derivatives are needed. Typical questions ask for a gradient, the equation of a tangent or normal, or the points where the tangent is horizontal.

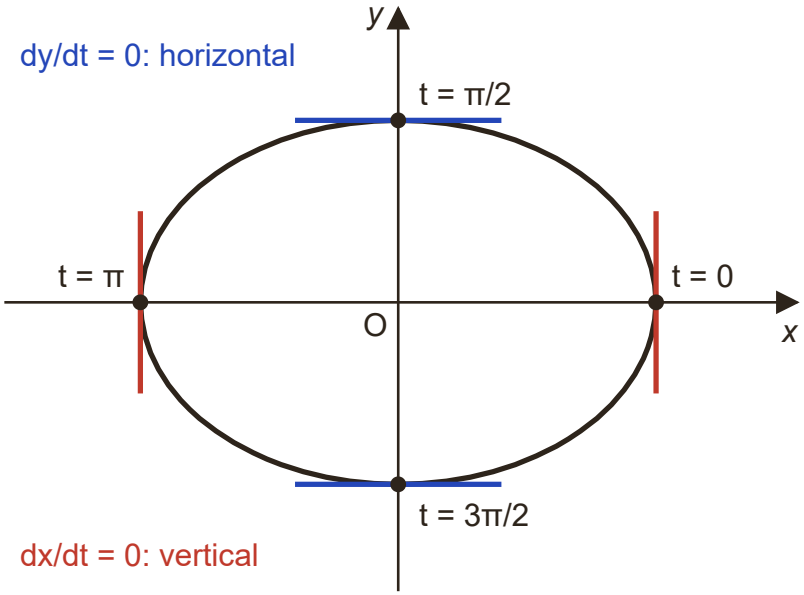
Key facts

- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}(xy) = x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y$
- Horizontal tangent: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$; vertical tangent: $\dfrac{\mathrm{d}x}{\mathrm{d}y} = 0$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}(y^n) = ny^{n-1}\dfrac{\mathrm{d}y}{\mathrm{d}x}$
- $\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^y) = \mathrm{e}^y\dfrac{\mathrm{d}y}{\mathrm{d}x}$; $\dfrac{\mathrm{d}}{\mathrm{d}x}(\ln y) = \dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x}$
- Tangent: $y - y_1 = m(x - x_1)$; normal gradient $-\dfrac{1}{m}$

Notes

Parametric differentiation

- If $x = f(t)$ and $y = g(t)$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$, provided $\frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$
- The answer is in terms of t. To find the gradient at a point, first find the value of t at that point.
- The tangent is horizontal where $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ and vertical where $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ (in each case the other derivative must not also be 0).



For $x = 3\cos t$, $y = 2\sin t$: horizontal tangents where $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$, vertical tangents where $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$

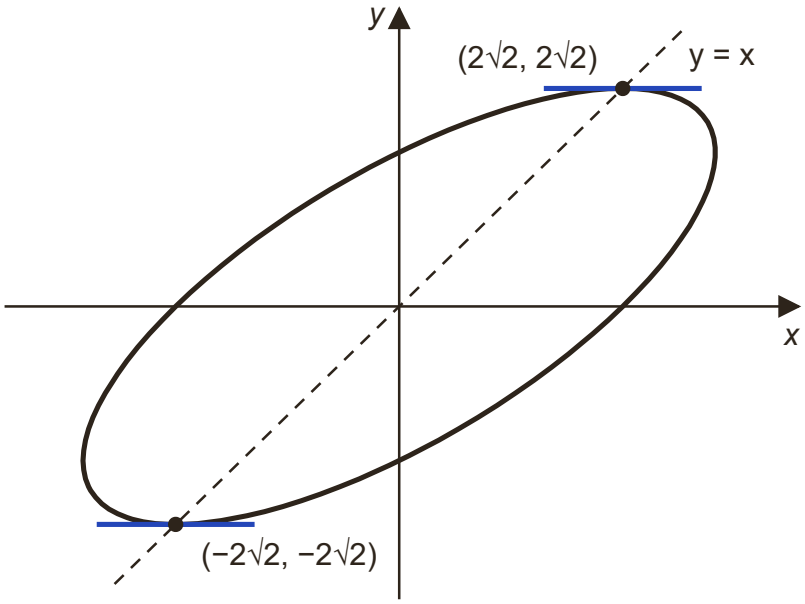
- For a tangent or normal, get the point from $x = f(t)$, $y = g(t)$ and the gradient from $\frac{\mathrm{d}y}{\mathrm{d}x}$

Implicit differentiation

- Differentiate every term with respect to x, on both sides of the equation.
- For a term in y, differentiate with respect to y, then multiply by $\frac{\mathrm{d}y}{\mathrm{d}x}$ (chain rule): $\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^y) = \mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}}{\mathrm{d}x}(\sin y) = \cos y\frac{\mathrm{d}y}{\mathrm{d}x}$
- For a product of x and y terms use the product rule: $\frac{\mathrm{d}}{\mathrm{d}x}(x^2y^3) = 2xy^3 + 3x^2y^2\frac{\mathrm{d}y}{\mathrm{d}x}$
- Constants differentiate to 0. Then collect the $\frac{\mathrm{d}y}{\mathrm{d}x}$ terms on one side, factorise and divide.

Using the derivative

- Gradient at a point: substitute both x and y. Check that the point lies on the curve first.
- Stationary points: set the numerator of $\frac{\mathrm{d}y}{\mathrm{d}x}$ equal to 0 to get a link between x and y, then substitute it into the curve's equation.



For $x^2 - 2xy + 2y^2 = 8$: $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ where $y = x$, which meets the curve at the top and bottom points.

- Only first derivatives of implicit and parametric curves are needed in this course.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Parametric: find $\frac{dy}{dx}$ in terms of t

3 to 4 marks | **Grade C**

1. Differentiate x and y separately with respect to t
2. Divide: $\frac{dy}{dt} \div \frac{dx}{dt}$
3. Simplify, using trig identities if useful, then substitute the value of t

Example. A curve has parametric equations $x = 3\cos t$, $y = 2\sin t$, $0 < t < \pi$.

(a) Find $\frac{dy}{dx}$ in terms of t.

(b) Find the gradient of the curve at the point where $t = \frac{\pi}{4}$

Model answer.

(a) $\frac{dx}{dt} = -3\sin t$, $\frac{dy}{dt} = 2\cos t$ B1

$$\frac{dy}{dx} = \frac{2\cos t}{-3\sin t} = -\frac{2}{3}\cot t$$ M1 A1

(b) $\cot\frac{\pi}{4} = 1$, so the gradient is $-\frac{2}{3}$ B1

Tangent to a parametric curve 5 to 6 marks **Grade B**

1. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t
2. Substitute the value of t to get the gradient and the coordinates of the point.
3. Use $y - y_1 = m(x - x_1)$ and rearrange into the form required.

Example. A curve C has parametric equations $x = t^2 + 1$, $y = t^3 - 3t$. Find an equation of the tangent to C at the point where $t = 2$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

Model answer.

$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 - 3$ B1

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3t^2 - 3}{2t}$ M1; at $t = 2$ the gradient is $\frac{9}{4}$ A1

At $t = 2$: $x = 5$, $y = 2$ B1

$y - 2 = \frac{9}{4}(x - 5)$ M1

$9x - 4y - 37 = 0$ A1

Implicit: find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and a gradient 5 to 6 marks **Grade B**

1. Differentiate each term with respect to x, using the product rule on xy terms.
2. Collect the $\frac{\mathrm{d}y}{\mathrm{d}x}$ terms on one side and factorise.
3. Divide to make $\frac{\mathrm{d}y}{\mathrm{d}x}$ the subject, then substitute the point.

Example. The curve C has equation $x^2 + 3xy - y^2 = 3$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

(b) Find the gradient of C at the point $(1, 2)$

Model answer.

(a) $2x + 3y + 3x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ M1 B1 A1

(M1 for a y term giving a $\frac{\mathrm{d}y}{\mathrm{d}x}$ term, B1 for the product rule on $3xy$, A1 all correct)

$\frac{\mathrm{d}y}{\mathrm{d}x}(3x - 2y) = -(2x + 3y)$, so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x + 3y}{2y - 3x}$ M1 A1

(b) At $(1, 2)$: $\frac{2 + 6}{4 - 3} = 8$ B1

Stationary points on an implicit curve

6 marks **Grade A**

1. Differentiate implicitly and make $\frac{\mathrm{d}y}{\mathrm{d}x}$ the subject.
2. Set the numerator equal to 0 to get a relationship such as $y = x$
3. Substitute into the equation of the curve and solve.
4. Give both coordinates of each point, and check the denominator is not 0 there.

Example. The curve C has equation $x^2 - 2xy + 2y^2 = 8$. Find the coordinates of the points on C where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

Model answer.

$2x - 2y - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ M1 A1

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y - 2x}{4y - 2x}$

$= \frac{y - x}{2y - x}$ A1

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow y = x$ M1

$x^2 - 2x^2 + 2x^2 = 8 \Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$ M1

$(2\sqrt{2}, 2\sqrt{2})$ and $(-2\sqrt{2}, -2\sqrt{2})$ A1

Don't lose marks

- Differentiating xy as just $x\frac{\mathrm{d}y}{\mathrm{d}x}$ or just y, missing half of the product rule.
- Differentiating a constant on the right-hand side to itself instead of 0.
- Putting the parametric derivatives upside down: it is $\frac{\mathrm{d}y}{\mathrm{d}t}$ over $\frac{\mathrm{d}x}{\mathrm{d}t}$
- Substituting an x-coordinate as if it were the value of t
- For stationary points, finding $y = x$ but not substituting back into the curve to get the actual points.

More tips

Memory tricks

- Every y term gets a $\frac{\mathrm{d}y}{\mathrm{d}x}$ tag when you differentiate it.
- Parametric: y on top, x underneath, both differentiated with respect to t
- Substitute the point into the curve's equation first: if it does not fit, you have misread something.
- Write the differentiated equation in full before rearranging, so no $\frac{\mathrm{d}y}{\mathrm{d}x}$ term gets lost.

Exam technique

- Write the fully differentiated equation before rearranging; most of the method marks are for this line.
- In 'show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \ldots$' questions, rearrange to exactly the given form (you can multiply top and bottom by -1).
- Give points as exact coordinates, such as $(2\sqrt{2}, 2\sqrt{2})$, unless told otherwise.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade C — **Differentiate parametric equations**

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$, giving an answer in terms of t

Grade C — **Differentiate terms in y implicitly**

Differentiate with respect to y and multiply by $\frac{\mathrm{d}y}{\mathrm{d}x}$: $\frac{\mathrm{d}}{\mathrm{d}x}(y^3) = 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$

Grade B — **Differentiate products such as xy implicitly**

Use the product rule: $\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x\frac{\mathrm{d}y}{\mathrm{d}x} + y$

Grade B — **Find tangents and normals to parametric curves**

Find the value of t, the point (x, y) and the gradient, then use $y - y_1 = m(x - x_1)$

Grade A — **Find stationary points on implicit curves**

Set the numerator of $\frac{\mathrm{d}y}{\mathrm{d}x}$ equal to 0, then solve together with the curve's equation.

Grade A* — **Find where tangents are parallel to an axis**

Horizontal: $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$; vertical: the denominator of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is 0.

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