Product, quotient and chain rules

P7.4a · Pure mathematics · revision notes

The chain rule for composite functions, the product rule and the quotient rule, and using $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \div \frac{\mathrm{d}x}{\mathrm{d}y}$ when x is given in terms of y. Questions often combine these with stationary points, tangents or 'show that' algebra, and can include the derivatives of sec, cosec and cot.

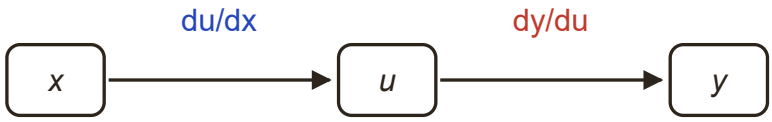
Key facts

- Chain rule: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$
- Quotient rule: $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$
- $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)]^n = n[f(x)]^{n-1}f'(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{f(x)} = f'(x)\mathrm{e}^{f(x)}$
- $\frac{\mathrm{d}}{\mathrm{d}x}\ln f(x) = \frac{f'(x)}{f(x)}$
- $\sec x \to \sec x \tan x$;
  $\operatorname{cosec} x \to -\operatorname{cosec} x \cot x$;
  $\cot x \to -\operatorname{cosec}^2 x$
- Product rule: $(uv)' = uv' + vu'$
- $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}$

Notes

The chain rule

- If y is a function of u and u is a function of x, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u} \times \dfrac{\mathrm{d}u}{\mathrm{d}x}$



dy/dx = dy/du × du/dx
(multiply the rates along the chain)

A small change in x changes u, which changes y: the rates multiply.

- In practice: differentiate the outside function (leaving the inside alone), then multiply by the derivative of the inside.
- $(ax+b)^n \to an(ax+b)^{n-1}$; $\mathrm{e}^{f(x)} \to f'(x)\mathrm{e}^{f(x)}$; $\ln f(x) \to \dfrac{f'(x)}{f(x)}$; $\sin f(x) \to f'(x)\cos f(x)$
- $\sin^2 x$ means $(\sin x)^2$, so its derivative is $2\sin x \cos x$

Product and quotient rules

- Product rule: if $y = uv$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = u\dfrac{\mathrm{d}v}{\mathrm{d}x} + v\dfrac{\mathrm{d}u}{\mathrm{d}x}$
- Quotient rule: if $y = \dfrac{u}{v}$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$. It is in the formula booklet; the order of the top matters.
- Write down u, v, u' and v' before you substitute.
- Factorise the result, taking out common factors such as e^{2x} or $(x+1)^3$; you need this to solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

When x is given in terms of y

- Find $\dfrac{\mathrm{d}x}{\mathrm{d}y}$, then use $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\mathrm{d}x/\mathrm{d}y}$
- E.g. $x = y^2 + 3y$ gives $\dfrac{\mathrm{d}x}{\mathrm{d}y} = 2y + 3$, so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{2y+3}$
- To give the answer in terms of x, use an identity, e.g. $\sec^2 y = 1 + \tan^2 y$

sec, cosec and cot

- Using the chain rule or the quotient rule: $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$, $\dfrac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\operatorname{cosec}^2 x$
- These, with their kx versions, are in the formula booklet, but you may be asked to prove them.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Differentiate using the chain rule

2 to 4 marks | Grade C

1. Identify the inside function and the outside function.
2. Differentiate the outside, keeping the inside unchanged.
3. Multiply by the derivative of the inside and simplify.

Example. Differentiate with respect to x:

(a) $\sqrt{1+2x^2}$

(b) $\ln(\cos x)$, $0 < x < \dfrac{\pi}{2}$

Model answer.

(a) $\dfrac{1}{2}(1+2x^2)^{-\frac{1}{2}} \times 4x$ M1 $= \dfrac{2x}{\sqrt{1+2x^2}}$ A1

(b) $\dfrac{-\sin x}{\cos x}$ M1 $= -\tan x$ A1

Product rule, then stationary points

5 to 6 marks | Grade B

1. Set u and v, and find u' and v'
2. Apply $uv' + vu'$
3. Factorise, including any exponential factor.
4. Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$, using the fact that an exponential is never 0.

Example. The curve C has equation $y = x^2\mathrm{e}^{-2x}$.

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, giving your answer in a fully factorised form.

(b) Hence find the exact coordinates of the stationary points of C

Model answer.

(a) $u = x^2$, $v = \mathrm{e}^{-2x}$, $u' = 2x$, $v' = -2\mathrm{e}^{-2x}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x\mathrm{e}^{-2x} - 2x^2\mathrm{e}^{-2x}$ M1 A1 $= 2x(1-x)\mathrm{e}^{-2x}$ A1

(b) $\mathrm{e}^{-2x} \neq 0$, so $x = 0$ or $x = 1$ M1

$(0, 0)$ and $(1, \mathrm{e}^{-2})$ A1

Quotient rule: show that, then use it

4 to 5 marks | **Grade B**

1. Write u, v, u' and v'
2. Substitute into $\dfrac{vu' - uv'}{v^2}$, keeping the order of the top.
3. Cancel common factors to reach the given answer.
4. Set the numerator equal to 0 to find stationary points.

Example. The curve C has equation $y = \dfrac{\ln x}{x^2}$, $x > 0$.

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1 - 2\ln x}{x^3}$.

(b) Find the exact coordinates of the stationary point of C

Model answer.

(a) $u = \ln x$, $v = x^2$, $u' = \dfrac{1}{x}$, $v' = 2x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 \times \frac{1}{x} - 2x\ln x}{x^4} \text{ M1 A1} = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3} \text{ A1}$$

(b) $1 - 2\ln x = 0 \Rightarrow x = \mathrm{e}^{\frac{1}{2}} = \sqrt{\mathrm{e}}$ M1

$y = \dfrac{1/2}{\mathrm{e}} = \dfrac{1}{2\mathrm{e}}$, so the point is $\left(\sqrt{\mathrm{e}}, \dfrac{1}{2\mathrm{e}}\right)$ A1

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when x is given in terms of y

3 to 4 marks | **Grade A**

1. Differentiate x with respect to y
2. Use an identity to write $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of x
3. Take the reciprocal.

Example. Given that $x = \tan 2y$, $-\dfrac{\pi}{4} < y < \dfrac{\pi}{4}$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x

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Model answer.

$$\frac{\mathrm{d}x}{\mathrm{d}y} = 2\sec^2 2y \quad \text{M1 A1}$$

$$= 2(1 + \tan^2 2y) = 2(1 + x^2) \quad \text{M1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2(1 + x^2)} \quad \text{A1}$$

Prove a derivative of sec, cosec or cot

3 marks | **Grade A**

1. Write the function as a power of $\sin x$ or $\cos x$, or as a quotient.
2. Differentiate with the chain rule (or the quotient rule).
3. Split the result into a product of two reciprocal trig functions.

Example. Prove that $\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

Model answer.

$\operatorname{cosec} x = (\sin x)^{-1}$, so $\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x) = -(\sin x)^{-2}\cos x$ M1 A1

$= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x}$

$= -\operatorname{cosec} x \cot x$ A1

Don't lose marks

- Swapping the order of the top in the quotient rule, which flips the sign.
- Forgetting to multiply by the derivative of the inside.
- Forgetting to square the denominator in the quotient rule.
- Dividing both sides by x instead of factorising, and so losing the solution $x = 0$ of $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
- Leaving $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of y when the question asks for it in terms of x

More tips

Memory tricks

- Chain rule: derivative of the outside times derivative of the inside.
- Quotient rule order: bottom times derivative of top, minus top times derivative of bottom, all over bottom squared.
- $\ln f(x)$ differentiates to 'derivative of the inside over the inside'.
- Exponentials are never 0: once you factor out $\mathrm{e}^{(\cdots)}$, only the other factors can make $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

Exam technique

- In 'show that' questions write u, v, u', v' and the unsimplified rule before simplifying.
- 'Hence' means use the answer you have just found, usually a factorised derivative.
- Give exact coordinates (e^{-2}, $\sqrt{\mathrm{e}}$) unless decimals are asked for.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade D **Use the chain rule on powers of functions**

E.g. $\frac{\mathrm{d}}{\mathrm{d}x}(3x-1)^5 = 15(3x-1)^4$

Grade C **Differentiate composite exponential, log and trig functions**

$\mathrm{e}^{f(x)} \to f'(x)\mathrm{e}^{f(x)}$, $\ln f(x) \to \frac{f'(x)}{f(x)}$, $\sin f(x) \to f'(x)\cos f(x)$

Grade C **Use the product rule**

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

Grade B **Use the quotient rule**

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$ (given in the formula booklet).

Grade B **Use** $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \div \frac{\mathrm{d}x}{\mathrm{d}y}$

When x is given in terms of y, differentiate with respect to y and take the reciprocal.

Grade A **Prove the derivatives of sec, cosec and cot**

Write e.g. $\sec x = (\cos x)^{-1}$ and use the chain rule or the quotient rule.

Grade A **Combine rules and factorise to find stationary points**

Factorise answers like $2x(1-x)\mathrm{e}^{-2x}$ so that $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ is easy to solve.

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