Tangents, normals and stationary points

P7.3 · Pure mathematics · revision notes

Using derivatives to find gradients, equations of tangents and normals, stationary points and their nature, and where a function is increasing or decreasing. It also covers optimisation: finding the maximum or minimum value of a quantity in a real problem. These are among the most common calculus questions on the Pure papers.

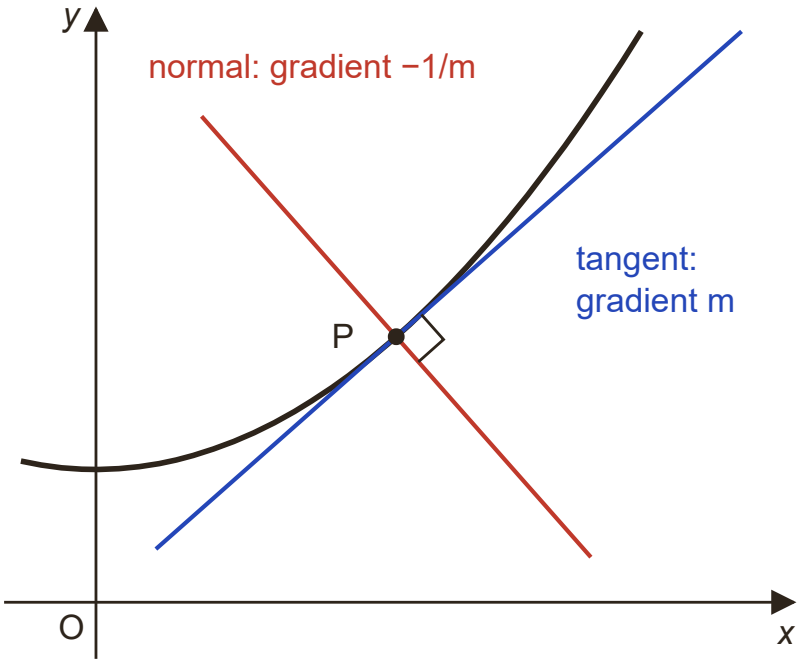
Key facts

- Tangent: $y - y_1 = m(x - x_1)$, where $m = f'(x_1)$
- Perpendicular lines: $m_1 m_2 = -1$
- $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$: minimum
- $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$: not decided, so check the sign of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ either side
- Point of inflection: $f''(x) = 0$ and $f''(x)$ changes sign
- Normal gradient $= -\dfrac{1}{m}$
- Stationary point: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$
- $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$: maximum
- Increasing: $f'(x) \geq 0$; decreasing: $f'(x) \leq 0$

Notes

Tangents and normals

- The gradient of the curve at $x = a$ is $f'(a)$
- Tangent at (x_1, y_1): $y - y_1 = m(x - x_1)$ with $m = f'(x_1)$
- Normal: perpendicular to the tangent, so its gradient is $-\dfrac{1}{m}$ (since $m_1 m_2 = -1$).

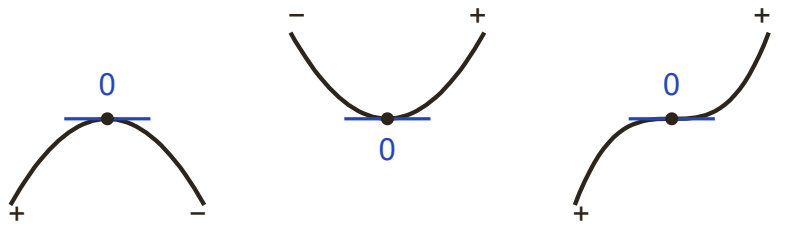


The normal is perpendicular to the tangent, so the product of their gradients is -1

- Find y_1 from the equation of the curve, never from $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

Stationary points

- At a stationary point $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. It can be a local maximum, a local minimum or a stationary point of inflection.
- If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ it is a minimum; if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ it is a maximum.
- If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ this test does not decide: check the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ just either side. The same sign on both sides means a stationary point of inflection.

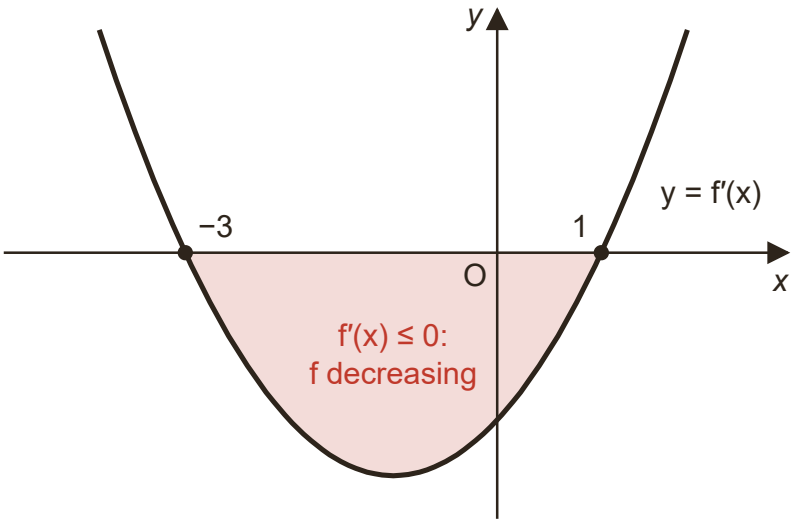


The signs show $\frac{\mathrm{d}y}{\mathrm{d}x}$ just before, at and just after each stationary point.

- A **point of inflection** is where the curve changes between convex and concave: $f''(x) = 0$ and $f''(x)$ changes sign there. It need not be stationary.
- To show a point is a point of inflection, find $f''(x)$ just either side of it and show the signs differ; $f''(x) = 0$ alone is not enough.

Increasing and decreasing functions

- f is increasing on an interval if $f'(x) \geq 0$ throughout it, and decreasing if $f'(x) \leq 0$. (Strictly increasing means $f'(x) > 0$.)
- For a cubic, $f'(x)$ is a quadratic: find its roots, draw a quick sketch of $y = f'(x)$ yourself, and read off where it is above or below the x-axis.



For $f(x) = x^3 + 3x^2 - 9x + 1$: $f'(x) \leq 0$ between the roots, so f is decreasing for $-3 \leq x \leq 1$

- To show a function is increasing for all x, write $f'(x)$ in a form that is clearly never negative, e.g. $3(x-1)^2 + 2$

Optimisation

- Write the quantity to be maximised or minimised in terms of one variable, using the given constraint to eliminate the other.
- Differentiate, set the derivative equal to 0 and solve.
- Justify the nature (usually with the second derivative), then answer the actual question, with units.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Find the equation of a normal or tangent 4 to 5 marks **Grade D**

1. Differentiate and substitute the x-coordinate to get the tangent gradient m
2. For a normal, use $-\dfrac{1}{m}$
3. Substitute into $y - y_1 = m(x - x_1)$
4. Rearrange into the form asked for, with integer coefficients.

Example. The curve C has equation $y = x^3 - 5x + 3$. The point $P(2, 1)$ lies on C. Find an equation of the normal to C at P, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

Model answer.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 5$ M1 A1

At $x = 2$ the gradient of the tangent is 7, so the gradient of the normal is $-\dfrac{1}{7}$ M1

$y - 1 = -\dfrac{1}{7}(x - 2)$ M1

$x + 7y - 9 = 0$ A1

Find and classify stationary points 5 to 6 marks **Grade C**

1. Differentiate and set $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$
2. Solve (factorise if you can) and find each y-coordinate from the original equation.
3. Find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and substitute each x-value to decide whether it is a maximum or a minimum.

Example. Find the coordinates of the stationary points on the curve with equation $y = 2x^3 - 9x^2 + 12x - 3$, and determine the nature of each.

Model answer.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 18x + 12$ M1 A1

$6(x - 1)(x - 2) = 0 \Rightarrow x = 1, 2$ M1

Points $(1, 2)$ and $(2, 1)$ A1

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x - 18$: at $x = 1$ it is $-6 < 0$, so $(1, 2)$ is a maximum; at $x = 2$ it is $6 > 0$, so $(2, 1)$ is a minimum M1 A1

Find where a function is increasing or decreasing

3 to 4 marks **Grade B**

1. Differentiate to get $f'(x)$
2. Solve $f'(x) = 0$ to find the critical values.
3. Decreasing means $f'(x) \leq 0$: for a quadratic with a positive x^2 term, that is between the roots.

Example. $f(x) = x^3 + 3x^2 - 9x + 1$. Find the set of values of x for which $f(x)$ is decreasing.

Model answer.

$f'(x) = 3x^2 + 6x - 9$	M1 A1
$3(x+3)(x-1) = 0 \Rightarrow x = -3, 1$	M1
Decreasing for $-3 \leq x \leq 1$, i.e. $\{x : -3 \leq x \leq 1\}$	A1

Optimisation 6 to 9 marks **Grade A**

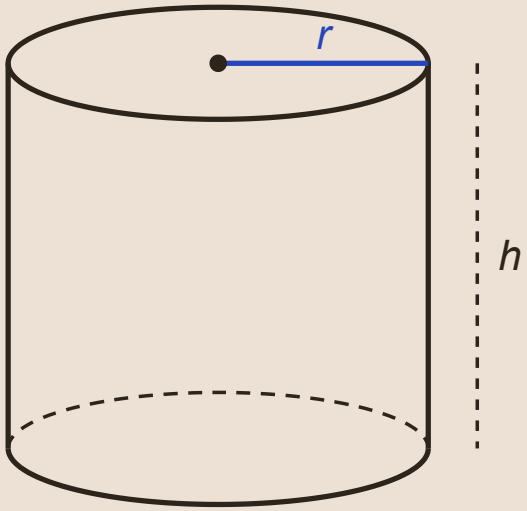
1. Use the constraint (here the volume) to write the quantity in terms of one variable.
2. Differentiate and set the derivative equal to 0.
3. Solve, then substitute back to find the optimum value.
4. Justify the nature with the second derivative and give units.

Example. A closed cylindrical can has radius r cm and height h cm. Its volume is 250π cm^3.

(a) Show that the total surface area, S cm^2, is given by $S = 2\pi r^2 + \dfrac{500\pi}{r}$.

(b) Use calculus to find the minimum value of S, giving your answer as an exact multiple of π.

(c) Justify that this value is a minimum.



Model answer.

(a) $\pi r^2 h = 250\pi \Rightarrow h = \dfrac{250}{r^2}$	M1
$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \dfrac{500\pi}{r}$	A1
(b) $\dfrac{\mathrm{d}S}{\mathrm{d}r} = 4\pi r - \dfrac{500\pi}{r^2}$	M1 A1
$4\pi r - \dfrac{500\pi}{r^2} = 0 \Rightarrow r^3 = 125 \Rightarrow r = 5$	M1 A1
$S = 50\pi + 100\pi = 150\pi$ cm^2	A1
(c) $\dfrac{\mathrm{d}^2 S}{\mathrm{d}r^2} = 4\pi + \dfrac{1000\pi}{r^3} = 12\pi > 0$ when $r = 5$, so it is a minimum	M1 A1

Don't lose marks

- Using the tangent gradient in the equation of the normal.

- Substituting into $\frac{\mathrm{d}y}{\mathrm{d}x}$ to find a y-coordinate.
- Not giving the line in the required form, e.g. leaving fractions when integers are asked for.
- Saying 'point of inflection' because $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, without testing further.
- In optimisation, finding r but not the maximum or minimum value asked for, or not justifying its nature.

More tips

Memory tricks

- Normal gradient: flip the tangent gradient and change its sign (7 becomes $-\frac{1}{7}$).
- Second derivative: positive means a U shape (minimum); negative means a ∩ shape (maximum).
- Use your calculator's table function to check the y-coordinates of stationary points.
- A cubic with a positive x^3 term goes up, down, up: it is decreasing between its two stationary points.

Exam technique

- 'Determine the nature' needs a reason: quote the value or sign of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at each point.
- 'Use calculus' means you must show the differentiation; an answer from a graph or trial and improvement gets no marks.
- In 'show that' parts of optimisation questions, show where each term comes from (e.g. the two circular ends and the curved surface).

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Find the gradient at a point**
Differentiate, then substitute the x-coordinate into $\frac{\mathrm{d}y}{\mathrm{d}x}$

Grade D — **Find the equation of a tangent**
Use $y - y_1 = m(x - x_1)$ with $m = f'(x_1)$

Grade D — **Find the equation of a normal**
The normal is perpendicular to the tangent, so its gradient is $-\frac{1}{m}$

Grade C — **Find stationary points**
Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, then find each y-coordinate from the curve's equation.

Grade C — **Classify stationary points**
$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$: minimum; $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$: maximum; if it is 0, test the gradient on each side.

Grade B — **Find where a function is increasing or decreasing**
Solve $f'(x) \geq 0$ (increasing) or $f'(x) \leq 0$ (decreasing).

Grade A — **Solve optimisation problems**
Form an expression in one variable, differentiate, solve, then justify the maximum or minimum.

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