Differentiating standard functions

P7.2 · Pure mathematics · revision notes

The standard derivatives: powers of x (including negative and fractional powers), e^{kx}, a^{kx}, $\ln x$, $\sin kx$, $\cos kx$ and $\tan kx$, with sums, differences and constant multiples. These appear in almost every calculus question, so they must be automatic.

Key facts

- $x^n \to nx^{n-1}$
- $a^{kx} \to ka^{kx}\ln a$
- $\sin kx \to k\cos kx$
- $\tan kx \to k\sec^2 kx$
- $\mathrm{e}^{kx} \to k\mathrm{e}^{kx}$
- $\ln x \to \frac{1}{x}$; $\ln kx \to \frac{1}{x}$
- $\cos kx \to -k\sin kx$
- Trig derivatives only work in radians

Notes

Powers of x

- $\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$ for any rational n. A constant differentiates to 0.
- Rewrite roots and fractions as powers first: $\sqrt{x} = x^{\frac{1}{2}}$, $\frac{1}{x^3} = x^{-3}$, $\frac{3}{2\sqrt{x}} = \frac{3}{2}x^{-\frac{1}{2}}$
- Expand brackets, and split fractions with a single term on the bottom: $\frac{x^3 - 4}{x^2} = x - 4x^{-2}$
- Differentiate sums, differences and constant multiples term by term. For example, $\frac{\mathrm{d}}{\mathrm{d}x}(kx) = k$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

Exponentials and logarithms

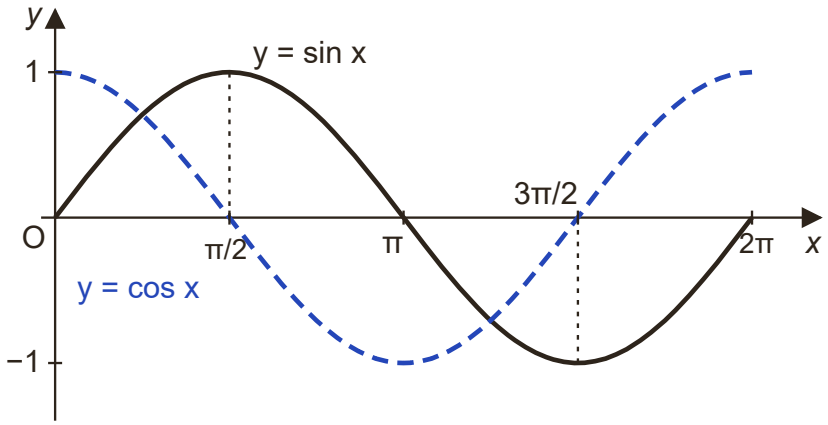
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{kx}) = k\mathrm{e}^{kx}$$

- $\frac{\mathrm{d}}{\mathrm{d}x}(a^{kx}) = ka^{kx}\ln a$, because $a^{kx} = \mathrm{e}^{kx\ln a}$
- $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$ for $x > 0$. Also $\frac{\mathrm{d}}{\mathrm{d}x}(\ln kx) = \frac{1}{x}$, since $\ln kx = \ln k + \ln x$ and $\ln k$ is a constant.
- Use the log laws first where they help: $\ln(x^5) = 5\ln x$, so its derivative is $\frac{5}{x}$

Trigonometric functions

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin kx) = k\cos kx$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos kx) = -k\sin kx$$



Where $\sin x$ is flat (at its max and min), $\cos x = 0$; where $\sin x$ is steepest, $\cos x = \pm 1$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan kx) = k\sec^2 kx$$

- These results are only true when x is in **radians**. Set your calculator to radians for gradient questions involving trig.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Differentiate expressions with powers of x

3 to 4 marks **Grade D**

1. Rewrite every term as a multiple of a power of x
2. Multiply by the power, then reduce the power by 1.
3. Tidy up; only write negative or fractional powers back as roots or fractions if asked.

Example. Given that $y = 4x^3 - 6\sqrt{x} + \dfrac{5}{x^2}$, $x > 0$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

Model answer.

$$y = 4x^3 - 6x^{\frac{1}{2}} + 5x^{-2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^2 - 3x^{-\frac{1}{2}} - 10x^{-3}$$

M1 for reducing any power by 1; A1 for two terms correct; A1 for all three terms correct

Differentiate exponentials, logs and trig functions

3 marks **Grade C**

1. Deal with each term separately using the standard results.
2. Multiply by k for e^{kx}, $\sin kx$ and $\cos kx$, and watch the sign for cos.
3. Remember that $\ln kx$ differentiates to $\dfrac{1}{x}$

Example. Given that $f(x) = 3\mathrm{e}^{2x} - 4\sin 5x + \ln 7x$, $x > 0$, find $f'(x)$

Model answer.

$$f'(x) = 6\mathrm{e}^{2x} - 20\cos 5x + \frac{1}{x}$$

B1 for $6\mathrm{e}^{2x}$; B1 for $-20\cos 5x$; B1 for $\dfrac{1}{x}$

Find an exact gradient involving a^x or $\tan kx$

4 marks | Grade B

1. Differentiate a^x as $a^x \ln a$ and $\tan kx$ as $k\sec^2 kx$
2. The curve crosses the y-axis where $x = 0$
3. Substitute, using $a^0 = 1$ and $\sec 0 = 1$, and leave the answer exact.

Example. The curve C has equation $y = 3^x + \tan 2x$, $-\frac{\pi}{4} < x < \frac{\pi}{4}$. Find the exact gradient of C at the point where it crosses the y-axis.

Model answer.

$$\frac{dy}{dx} = 3^x \ln 3 + 2\sec^2 2x$$ M1 A1 A1

(one A1 for each term)

At $x = 0$: gradient $= \ln 3 + 2$ A1

Simplify first, then differentiate (show that)

4 to 5 marks | Grade A

1. Write the denominator as a power: $x\sqrt{x} = x^{\frac{3}{2}}$
2. Split into separate terms and simplify each power.
3. Differentiate term by term.
4. Combine over a common denominator to reach the given form exactly.

Example. Given that $f(x) = \dfrac{2x^2 - 3}{x\sqrt{x}}$, $x > 0$, show that $f'(x) = \dfrac{2x^2 + 9}{2x^2\sqrt{x}}$

Model answer.

$$f(x)$$
$$= \frac{2x^2}{x^{\frac{3}{2}}} - \frac{3}{x^{\frac{3}{2}}}$$
$$= 2x^{\frac{1}{2}} - 3x^{-\frac{3}{2}}$$ M1 A1

$$f'(x) = x^{-\frac{1}{2}} + \frac{9}{2}x^{-\frac{5}{2}}$$ M1 A1

$$= \frac{2x^2 + 9}{2x^{\frac{5}{2}}}$$
$$= \frac{2x^2 + 9}{2x^2\sqrt{x}}$$ A1

Don't lose marks

- Working in degrees: the trig derivatives need radians.

- Getting the sign wrong: the derivative of $\cos x$ is $-\sin x$
- Using the power rule on e^x or a^x
- Writing $\frac{\mathrm{d}}{\mathrm{d}x}(\ln 2x) = \frac{2}{x}$ or $\frac{1}{2x}$ instead of $\frac{1}{x}$
- Rewriting $\frac{1}{2x^3}$ as $2x^{-3}$ instead of $\frac{1}{2}x^{-3}$
- Differentiating the top and the bottom of a fraction separately.

More tips

Memory tricks

- Power rule: multiply by the power, then take one off the power.
- Trig cycle: $\sin \to \cos \to -\sin \to -\cos \to \sin$. Each arrow is one differentiation.
- $\ln kx$: the k vanishes and the answer is always $\frac{1}{x}$
- Check a numerical gradient with your calculator's $\frac{\mathrm{d}}{\mathrm{d}x}$ function, if it has one.

Exam technique

- Show the line where you rewrite terms as powers; it often earns a method mark on its own.
- In 'show that' questions, finish with exactly the expression given.
- Use exact values ($\ln 3$, π, surds) when the question asks for an exact gradient.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Differentiate polynomials term by term**

$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$; a constant differentiates to 0.

Grade D — **Differentiate roots and reciprocals of x**

Rewrite as powers first: $\sqrt{x} = x^{\frac{1}{2}}$, $\frac{5}{x^2} = 5x^{-2}$

Grade C — **Differentiate e^{kx}, $\sin kx$ and $\cos kx$**

$k\mathrm{e}^{kx}$, $k\cos kx$ and $-k\sin kx$, with x in radians.

Grade C — **Differentiate $\ln x$ and $\ln kx$**

Both give $\frac{1}{x}$, because $\ln kx = \ln k + \ln x$

Grade B — **Differentiate a^{kx} and $\tan kx$**

$\frac{\mathrm{d}}{\mathrm{d}x}(a^{kx}) = ka^{kx}\ln a$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\tan kx) = k\sec^2 kx$

Grade A — **Simplify an expression before differentiating**

Split fractions, expand brackets and use log laws, e.g. $\ln x^3 = 3\ln x$

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