First principles, second derivatives and concavity

P7.1 · Pure mathematics · revision notes

What a derivative is: the gradient of the tangent, found as the limit of the gradient of a chord. You need to differentiate simple powers of x, and $\sin x$ and $\cos x$, from first principles, sketch gradient functions, and use the second derivative to find where a curve is convex or concave and where it has points of inflection.

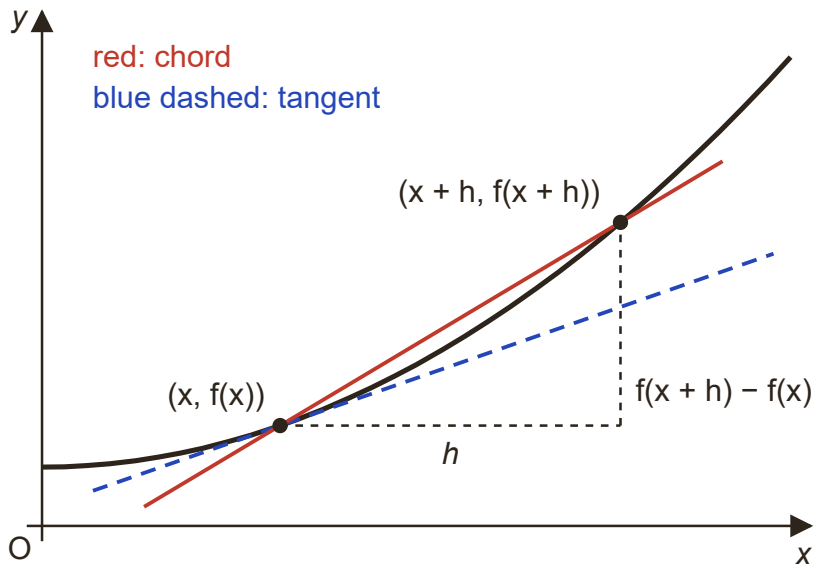
Key facts

- $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$
- $(x+h)^2 = x^2 + 2xh + h^2$
- $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$
- $\sin(A+B) = \sin A \cos B + \cos A \sin B$
- $\cos(A+B) = \cos A \cos B - \sin A \sin B$
- As $h \to 0$: $\dfrac{\sin h}{h} \to 1$ and $\dfrac{\cos h - 1}{h} \to 0$ (radians)
- Convex: $f''(x) \geq 0$; concave: $f''(x) \leq 0$
- Point of inflection: $f''(x) = 0$ and $f''(x)$ changes sign

Notes

The derivative as a limit

- The chord from $(x, f(x))$ to $(x+h, f(x+h))$ has gradient $\dfrac{f(x+h)-f(x)}{h}$
- As $h \to 0$ the chord becomes the tangent, so $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

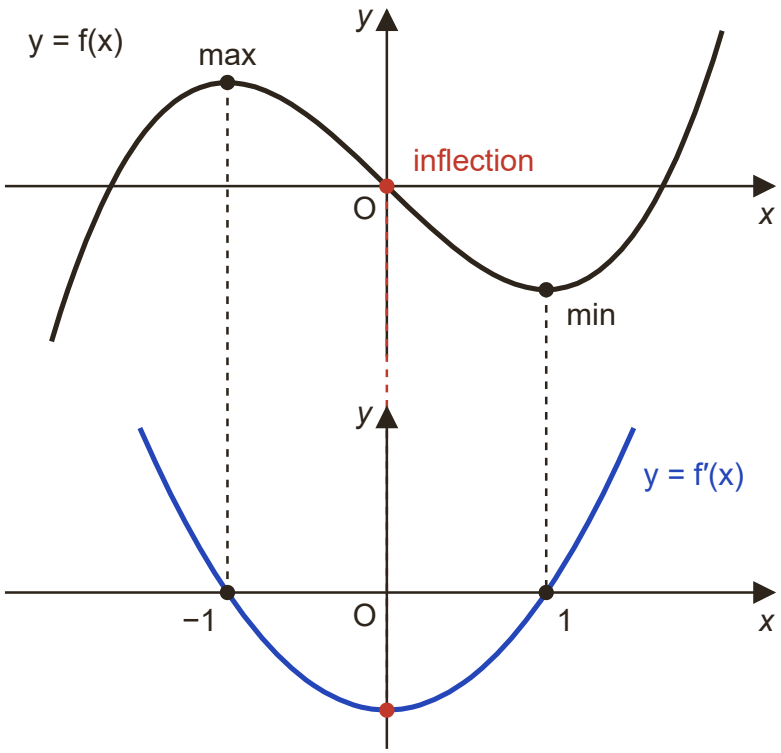


As $h \to 0$ the chord (red) turns into the tangent (blue), so its gradient tends to $f'(x)$

- $f'(x)$, or $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, is the gradient of the tangent at a general point, and the rate of change of y with respect to x
- The second derivative $f''(x)$, or $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, is found by differentiating $f'(x)$ again. It is the rate of change of the gradient.
- First principles method: write the limit, expand $f(x+h)$, cancel, divide every term by h, and only then let $h \to 0$
- For $\sin x$ and $\cos x$ (in radians): expand $\sin(x+h)$ or $\cos(x+h)$ with the addition formulae and use $\dfrac{\sin h}{h} \to 1$ and $\dfrac{\cos h - 1}{h} \to 0$ as $h \to 0$

Sketching the gradient function

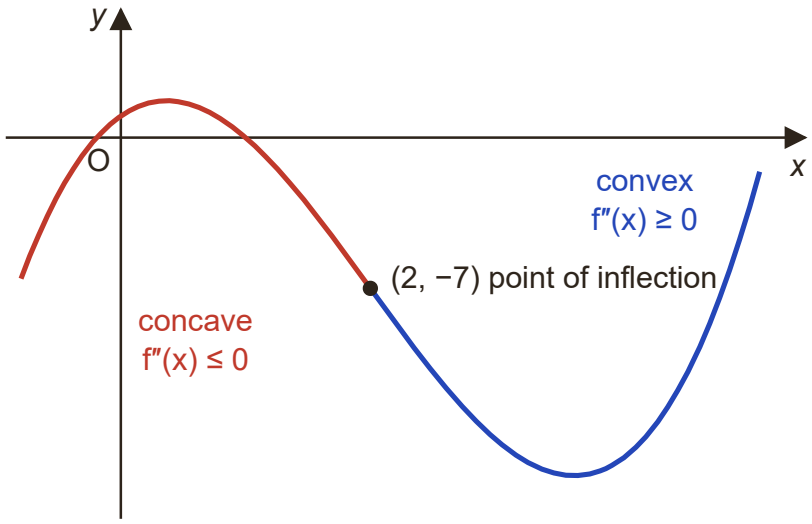
- Where $y = f(x)$ has a stationary point, the graph of $y = f'(x)$ meets the x-axis.
- Where $f(x)$ is increasing, $f'(x)$ is above the x-axis; where it is decreasing, below.
- Where $f(x)$ has a point of inflection, $f'(x)$ has a turning point.



Stationary points of f line up with the roots of f'; the point of inflection lines up with the turning point of f'

Second derivatives, convex and concave

- A curve is **convex** on an interval if $f''(x) \geq 0$ there (the gradient is increasing and the curve bends like a U), and **concave** if $f''(x) \leq 0$ (the gradient is decreasing and it bends like a ∩).
- A **point of inflection** is where the curve changes from convex to concave, or from concave to convex. There $f''(x) = 0$ and $f''(x)$ changes sign.



The curve bends like ∩ (concave) until $x = 2$, then like U (convex): $f''(x)$ changes sign there.

- $f''(x) = 0$ alone is not enough: for $y = x^4$, $f''(0) = 0$, but the curve is convex on both sides of $x = 0$, so there is no point of inflection.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Prove from first principles (powers of x)

4 to 5 marks | **Grade B**

1. Write the gradient of the chord: $\dfrac{f(x+h)-f(x)}{h}$
2. Expand $f(x+h)$ and cancel the terms without h
3. Divide every remaining term by h
4. Let $h \to 0$, using limit notation, and state the result.

Example. Prove, from first principles, that the derivative of $2x^3$ is $6x^2$

Model answer.

$f'(x) = \lim_{h\to 0} \dfrac{2(x+h)^3 - 2x^3}{h}$ B1

$= \lim\limits_{h\to 0} \dfrac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - 2x^3}{h}$ M1

$= \lim_{h\to 0}(6x^2 + 6xh + 2h^2)$ A1

As $h \to 0$, $6xh + 2h^2 \to 0$, so the derivative of $2x^3$ is $6x^2$ A1

Prove from first principles ($\sin x$ or $\cos x$)

5 marks | **Grade A***

1. Write the limit of $\dfrac{\cos(x+h)-\cos x}{h}$ (or the sine version).
2. Expand with the addition formula.
3. Group into a multiple of $\dfrac{\cos h - 1}{h}$ and a multiple of $\dfrac{\sin h}{h}$
4. Use the given limits and state the result.

Example. Given that x is measured in radians, prove from first principles that the derivative of $\cos x$ is $-\sin x$. You may assume that, as $h \to 0$, $\dfrac{\sin h}{h} \to 1$ and $\dfrac{\cos h - 1}{h} \to 0$

Model answer.

$f'(x) = \lim_{h\to 0} \dfrac{\cos(x+h) - \cos x}{h}$ B1

$= \lim_{h\to 0} \dfrac{\cos x \cos h - \sin x \sin h - \cos x}{h}$ M1 A1

$= \lim_{h\to 0} \left[\cos x \left(\dfrac{\cos h - 1}{h}\right) - \sin x \left(\dfrac{\sin h}{h}\right)\right]$ M1

$= \cos x \times 0 - \sin x \times 1 = -\sin x$ A1

Find where a curve is convex or concave

4 to 5 marks | **Grade B**

1. Differentiate twice.
2. Solve the inequality $f''(x) \leq 0$ (concave) or $f''(x) \geq 0$ (convex).
3. For a point of inflection, find where $f''(x) = 0$, check the sign change and find the y-coordinate from the curve's equation.

Example. The curve C has equation $y = x^3 - 6x^2 + 4x + 1$.
(a) Find the set of values of x for which C is concave.
(b) Find the coordinates of the point of inflection on C

Model answer.

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 12x + 4$ M1, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x - 12$ A1

Concave when $6x - 12 \leq 0$ M1, i.e. $x \leq 2$ A1

(b) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ changes sign at $x = 2$ (negative before, positive after);

$y = 8 - 24 + 8 + 1 = -7$, so $(2, -7)$ B1

Show that a curve has points of inflection

5 marks | **Grade A**

1. Find $f''(x)$ and solve $f''(x) = 0$
2. Test the sign of $f''(x)$ on each side of every root.
3. Only roots where the sign changes are points of inflection; find their coordinates.

Example. The curve C has equation $y = x^4 - 4x^3 + 10$. Show that C has exactly two points of inflection, and find their coordinates.

Model answer.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 12x^2$ M1, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 24x$ A1

$12x(x - 2) = 0 \Rightarrow x = 0$ or $x = 2$ M1

At $x = -1$, 1 and 3 the second derivative is 36, -12 and 36, so it changes sign at both roots M1

Points of inflection at $(0, 10)$ and $(2, -6)$ A1

Don't lose marks

- Putting $h = 0$ before dividing by h, which gives $\dfrac{0}{0}$
- Dropping the 'lim' notation, or writing '$h = 0$' instead of '$h \to 0$': the final mark is often lost.
- Using the rule nx^{n-1} in a first principles question: this scores no marks.

- Claiming a point of inflection just because $f''(x) = 0$, without checking that $f''(x)$ changes sign.
- Mixing up convex and concave.

More tips

Memory tricks

- Convex looks like a cup (f'' positive). Concave looks like the roof of a cave (f'' negative).
- First principles layout: limit, expand, cancel, divide by h, let $h \to 0$
- Differentiating lowers the degree by one: a cubic curve has a quadratic gradient function, so its gradient graph is a parabola.
- Point of inflection check: substitute a value just below and a value just above into $f''(x)$

Exam technique

- Write $\lim_{h \to 0}$ on every line until you actually take the limit, and finish with a concluding sentence.
- Give intervals as inequalities, or in set notation if the question asks, e.g. $\{x : x \leq 2\}$
- For points of inflection, show the sign test in writing; a small table of values of $f''(x)$ is a clear way to do it.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Interpret a derivative as gradient and rate**

$f'(x)$ is the gradient of the tangent at $(x, f(x))$ and the rate of change of y with respect to x

Grade D — **Find a second derivative**

Differentiate $\frac{\mathrm{d}y}{\mathrm{d}x}$ again to get $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, the rate of change of the gradient.

Grade C — **Sketch the gradient function of a curve**

Where the curve has a stationary point, $y = f'(x)$ meets the x-axis; where the curve increases, $f'(x) > 0$

Grade B — **Differentiate x^2 or x^3 from first principles**

Expand $f(x + h)$, simplify $\frac{f(x+h) - f(x)}{h}$, then let $h \to 0$

Grade B — **Find where a curve is convex or concave**

Convex where $f''(x) \geq 0$; concave where $f''(x) \leq 0$

Grade A — **Find and justify points of inflection**

Solve $f''(x) = 0$ and show that $f''(x)$ changes sign there.

Grade A* — **Differentiate $\sin x$ or $\cos x$ from first principles**

Use the addition formulae with $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$ as $h \to 0$

Free notes, Full packs, practice papers and mock exams at chhetriacademy.com. Written for this site; not taken from past papers. Stuck? 1-to-1 tutoring with a free 30-minute trial lesson.