Exponential growth and decay models

P6.7 · Pure mathematics · revision notes

Using $N = N_0\mathrm{e}^{kt}$ and related models for growth and decay, such as populations, radioactive decay, drug concentration and continuous compound interest. Questions ask you to find values, times and constants, to find rates of change, and to comment on the limitations of the model.

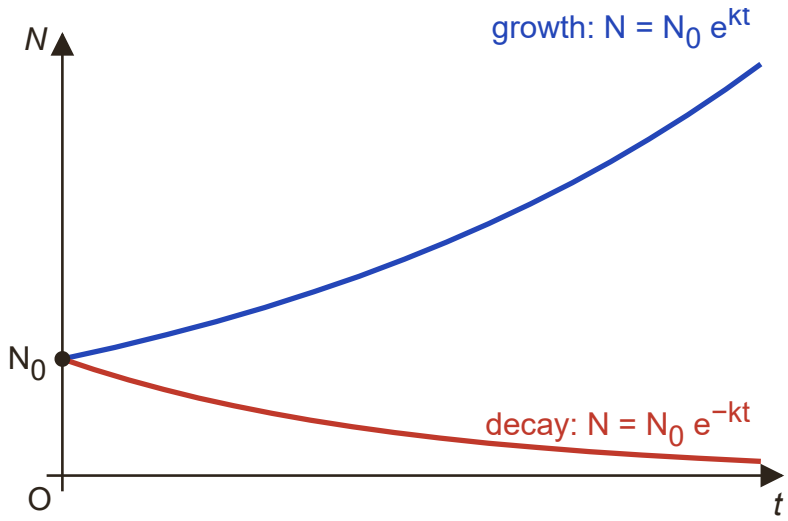
Key facts

- Growth: $N = N_0\mathrm{e}^{kt}$; decay: $N = N_0\mathrm{e}^{-kt}$ ($k > 0$)
- $\dfrac{\mathrm{d}N}{\mathrm{d}t} = kN$: rate proportional to amount
- $N = N_0b^t$ with $b = \mathrm{e}^k$: b is the multiplier per unit time
- $\theta = c + A\mathrm{e}^{-kt}$ tends to c as $t \to \infty$
- Initial value: put $t = 0$, giving N_0
- Half-life T: $k = \dfrac{\ln 2}{T}$
- Continuous compound interest: $A = P\mathrm{e}^{rt}$

Notes

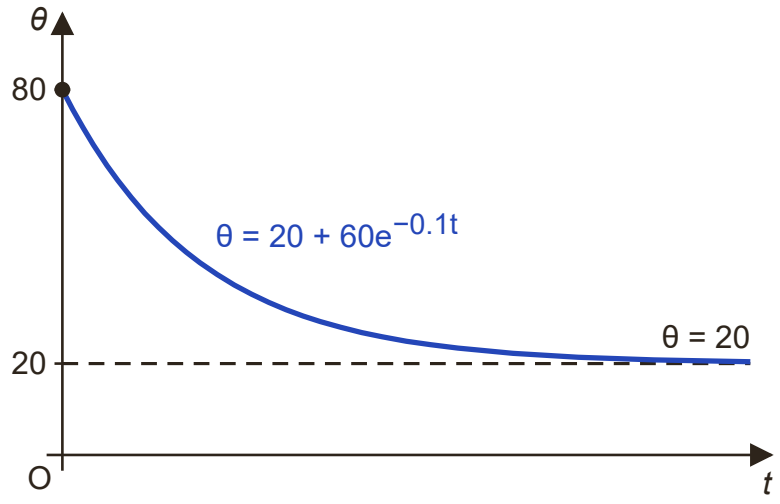
Exponential models

- Growth: $N = N_0 e^{kt}$ with $k > 0$. Decay: $N = N_0 e^{-kt}$ with $k > 0$



Growth $N = N_0 e^{kt}$ and decay $N = N_0 e^{-kt}$ both start at N_0 when $t = 0$

- N_0 is the initial value (when $t = 0$) and k is the rate constant.
- A model can also be written $N = N_0 b^t$, where b is the multiplier per unit of time ($b = e^k$). For example, 4% growth per year gives $b = 1.04$
- Continuous compound interest: an amount P invested at a rate r per year (as a decimal), compounded continuously, grows to Pe^{rt} after t years.
- A model with a constant term, such as $\theta = 20 + 60e^{-0.1t}$, tends to 20 as $t \to \infty$, like a cup of tea cooling to room temperature.



The constant term is the long-term value: the tea cools towards room temperature, 20 °C.

Solving model problems

- Value at a given time: substitute t
- Time to reach a value: rearrange to $\mathrm{e}^{kt} = \ldots$, then take ln.
- Unknown constants: use $t = 0$ to find N_0, then another data point to find k. If the half-life is T, then $\frac{1}{2} = \mathrm{e}^{-kT}$, so $k = \frac{\ln 2}{T}$
- Rate of change: differentiate. For $N = N_0\mathrm{e}^{kt}$, $\frac{\mathrm{d}N}{\mathrm{d}t} = kN_0\mathrm{e}^{kt} = kN$

Limitations and refinements

- Growth models predict growth without limit; in reality food, space or money run out.
- Decay models never quite reach 0, but real quantities such as a number of animals are whole numbers and can reach 0.
- A model based on data over a short time may not be valid far outside that time (extrapolation).
- Refinements: include a limiting value (e.g. $P = 2000 - 1500\mathrm{e}^{-kt}$ levels off at 2000), or let k change over time.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Use a model: initial value, later value, time taken 5 to 7 marks **Grade C**

1. Initial value: put $t = 0$
2. Later value: substitute the given t and give units.
3. Time taken: make e^{kt} the subject, take ln, then divide.
4. Round only at the end and answer in context.

Example. The mass, m grams, of a radioactive substance t days after measurements began is modelled by $m = 80\mathrm{e}^{-0.05t}$.
(a) State the initial mass of the substance.
(b) Find the mass after 10 days.
(c) Find the time taken for the mass to fall to 20 grams.

Model answer.

(a) 80 grams B1

(b) $m = 80\mathrm{e}^{-0.5}$ M1 $= 48.5$ grams (3 s.f.) A1

(c) $80\mathrm{e}^{-0.05t} = 20 \Rightarrow \mathrm{e}^{-0.05t} = 0.25$ M1

$-0.05t = \ln 0.25 \Rightarrow t = \frac{\ln 4}{0.05}$ M1

$t = 27.7$ days (3 s.f.) A1

Find the constants from data, then predict

4 to 6 marks | **Grade B**

1. Use the value at $t = 0$ to find the constant in front.
2. Substitute a second data point and solve for k using ln.
3. Use the full model to answer the question, keeping the unrounded k in your calculator.
4. Convert the time into a year or date if asked.

Example. The number of birds, P, in a colony t years after 1 January 2020 is modelled by $P = A\mathrm{e}^{kt}$, where A and k are constants. On 1 January 2020 there were 500 birds, and on 1 January 2024 there were 740 birds.
(a) Find the value of A and the value of k, giving k to 3 significant figures.
(b) Use the model to find the year in which the number of birds first exceeds 1500.

Model answer.

(a) $A = 500$ — B1

$740 = 500\mathrm{e}^{4k} \Rightarrow \mathrm{e}^{4k} = 1.48$ — M1

$k = \dfrac{1}{4}\ln 1.48$
$= 0.0980$ (3 s.f.) — A1

(b) $500\mathrm{e}^{kt} = 1500 \Rightarrow t = \dfrac{\ln 3}{k} = 11.2$ — M1

This is 11.2 years after 1 January 2020, so the number first exceeds 1500 during 2031 — A1

Find and interpret a rate of change

3 marks | **Grade A**

1. Differentiate the model with respect to t
2. Substitute the given time.
3. Interpret the sign: a negative derivative means the quantity is decreasing. Give units.

Example. The concentration, C mg per litre, of a drug in a patient's blood t hours after an injection is modelled by $C = 12\mathrm{e}^{-0.3t}$. Find the rate at which the concentration is decreasing 2 hours after the injection.

Model answer.

$\dfrac{\mathrm{d}C}{\mathrm{d}t} = -3.6\mathrm{e}^{-0.3t}$ — M1 A1

At $t = 2$: $\dfrac{\mathrm{d}C}{\mathrm{d}t} = -3.6\mathrm{e}^{-0.6} = -1.98$, so the concentration is decreasing at 1.98 mg per litre per hour — A1

Criticise and refine a model 1 to 2 marks **Grade C**

1. Say what the model predicts in the long run (e.g. unlimited growth, or never reaching 0).
2. Explain why that is unrealistic in this context.
3. Suggest a specific change, such as a limiting value.

Example. Give one limitation of the model for the number of birds in the colony above, and suggest how the model could be refined.

Model answer.

The model predicts that the number of birds grows without limit, but in reality food and nesting space are limited B1

Refine it so that the number of birds levels off at a maximum value, for example $P = M - B\mathrm{e}^{-kt}$ for positive constants M, B and k B1

Don't lose marks

- Rounding k too early, which can move a time into the wrong year.
- Converting a time into a year wrongly: 11.2 years after 1 January 2020 is during 2031, not 2032.
- Writing 'decreasing at -1.98': the derivative is -1.98, so the rate of decrease is 1.98.
- Using an average rate (change ÷ time) when the question wants the rate at an instant, which needs differentiation.
- Giving a limitation that is not about the context, e.g. just 'it is only a model'.

More tips

Memory tricks

- Half-life shortcut: $k = \dfrac{\ln 2}{\text{half-life}}$
- Always check that $t = 0$ gives the starting value you were told.
- Sense check: in a decay model later values must be smaller than the start; in a growth model, larger.
- Store k in your calculator's memory so you never re-type a rounded value.

Exam technique

- Limitations and interpretations must refer to the context (birds, money, the drug, the temperature), not just to the maths.
- Keep full calculator values through a question and round only the final answer.
- Include units: grams, days, mg per litre per hour.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E **Find the initial value of a model**
Put $t = 0$: $Ae^0 = A$, so A is the starting amount.

Grade D — **Use a model to predict a value**
Substitute the given time into the model and include units.

Grade C — **Find when a quantity reaches a value**
Rearrange to $e^{kt} = \ldots$, then take ln of both sides.

Grade C — **Give a limitation of a model**
E.g. a growth model predicts that a population increases without limit, which is unrealistic.

Grade B — **Find the constants from given data**
Use the starting value for A, then a second value to find k using ln.

Grade A — **Find and interpret a rate of change**
Differentiate the model and substitute; say whether the quantity is increasing or decreasing, with units.

Grade A* — **Refine a model and describe long-term behaviour**
E.g. $\theta = 20 + 60e^{-0.1t}$ tends to 20 as $t \to \infty$: the temperature of the room.

Free notes, Full packs, practice papers and mock exams at chhetriacademy.com. Written for this site; not taken from past papers. Stuck? 1-to-1 tutoring with a free 30-minute trial lesson.