Using log graphs to estimate parameters

P6.6 · Pure mathematics · revision notes

Taking logarithms turns the curves $y = ax^n$ and $y = kb^x$ into straight lines, so you can find the constants from the gradient and intercept of a line. Questions give data or a line of best fit, often in a real context, and ask you to find and interpret the constants.

Key facts

- $y = ax^n \Rightarrow \log y = n \log x + \log a$
- $y = kb^x \Rightarrow \log y = x \log b + \log k$
- Undo $\log_{10}$ with $10^{(\cdots)}$; undo ln with $\mathrm{e}^{(\cdots)}$
- $\log y$ against $\log x$: gradient n, intercept $\log a$
- $\log y$ against x: gradient $\log b$, intercept $\log k$
- Straight line for $\log y$ against $\log x$: power law. Straight line for $\log y$ against x: exponential

Notes

Power laws: $y = ax^n$

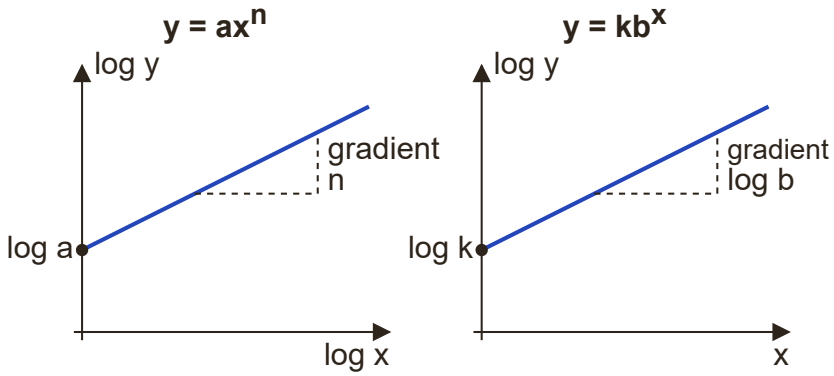
- Take logs of both sides (base 10 or base e): $\log y = \log a + n \log x$
- Compare with $Y = mX + c$, where $Y = \log y$ and $X = \log x$: the gradient m is n and the intercept c is $\log a$
- So if $y = ax^n$, the graph of $\log y$ against $\log x$ is a straight line.

Exponential laws: $y = kb^x$

- Take logs: $\log y = \log k + x \log b$
- Here $Y = \log y$ and $X = x$ (not $\log x$): the gradient is $\log b$ and the intercept is $\log k$
- So if $y = kb^x$, the graph of $\log y$ against x is a straight line.

Finding and using the constants

- Find the gradient from two points on the line (points on a line of best fit need not be data points).



Left: a power law gives a straight line of $\log y$ against $\log x$. Right: an exponential law gives a straight line of $\log y$ against x

- The intercept is the value of $\log y$ where the horizontal variable is 0. Check that the horizontal axis really starts at 0.
- Undo the logs: with $\log_{10}$, $a = 10^c$ and $b = 10^m$; with ln, $a = \mathrm{e}^c$ and $b = \mathrm{e}^m$. For a power law, n is the gradient itself.
- If you are given the equation of the line, e.g. $\log_{10} V = 4.3 - 0.1t$, read off m and c directly.
- In context, k is the starting value (when $x = 0$) and b is the multiplier for each increase of 1 in x. Predictions outside the range of the data (extrapolation) are unreliable.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Exponential model from a log graph 5 to 7 marks **Grade B**

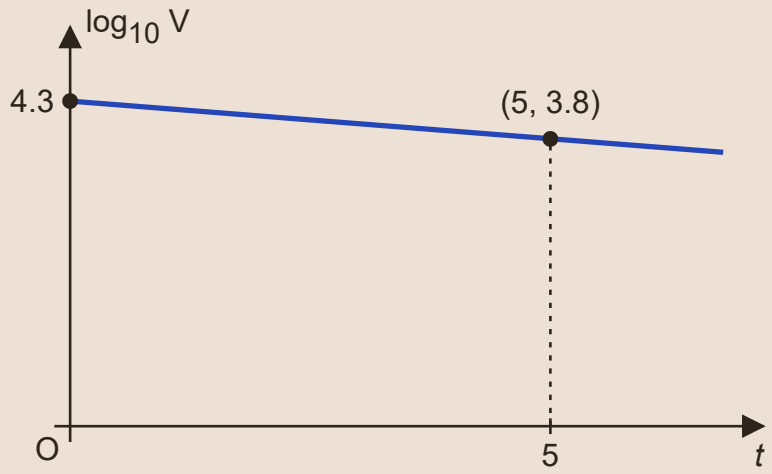
1. Take logs of the model to show it gives a straight line: $\log V = \log k + t \log b$
2. Find the gradient and the intercept of the line.
3. Undo the logs: $k = 10^{\text{intercept}}$ and $b = 10^{\text{gradient}}$
4. Interpret k and b in context.

Example. The value, V pounds, of a car t years after it was bought is modelled by $V = kb^t$, where k and b are constants. The graph of $\log_{10} V$ against t is a straight line through the points $(0, 4.3)$ and $(5, 3.8)$.

(a) Show that $\log_{10} V = \log_{10} k + t \log_{10} b$.

(b) Find the values of k and b, giving each to 3 significant figures.

(c) Interpret the value of b in this context.



Model answer.

(a) $\log_{10} V = \log_{10} k + \log_{10} b^t$
$= \log_{10} k + t \log_{10} b$ B1

(b) Gradient $= \dfrac{3.8 - 4.3}{5} = -0.1$ M1

$\log_{10} k = 4.3 \Rightarrow k = 10^{4.3} = 20\,000$ (3 s.f.) M1 A1

$\log_{10} b = -0.1 \Rightarrow b = 10^{-0.1} = 0.794$ (3 s.f.) A1

(c) Each year the car's value is multiplied by 0.794, so it loses about 20.6% of its value each year B1

Power law from a log–log graph

4 to 6 marks | Grade B

1. Take logs: $\ln y = n \ln x + \ln a$, so the gradient is n and the intercept is $\ln a$
2. Work out the gradient from two points on the line.
3. Find the intercept by substituting a point into $\ln y = n \ln x + c$
4. Undo the log, $a = \mathrm{e}^{c}$, then use the model.

Example. Two quantities x and y are thought to satisfy $y = ax^{n}$, where a and n are constants. A graph of $\ln y$ against $\ln x$ is a straight line through $(1, 3.5)$ and $(3, 2.5)$.
(a) Find the value of n and the value of a, giving a to 3 significant figures.
(b) Use the model to estimate y when $x = 9$

Model answer.

(a) $\ln y = n \ln x + \ln a$; gradient $n = \dfrac{2.5 - 3.5}{3 - 1}$

$= -0.5$ M1 A1

$3.5 = -0.5(1) + \ln a \Rightarrow \ln a = 4$ M1

$a = \mathrm{e}^{4} = 54.6$ A1

(b) $y = \mathrm{e}^{4} \times 9^{-0.5} = \dfrac{\mathrm{e}^{4}}{3}$ M1 $= 18.2$ A1

Explain how to test a model with a graph

3 to 4 marks | Grade C

1. Say which quantities to plot: $\log y$ against $\log x$ for a power law, or $\log y$ against x for an exponential.
2. Say that the model fits if the points lie close to a straight line.
3. Say how the gradient and the intercept give the constants.

Example. A scientist thinks that two variables P and t are related by $P = at^{n}$, where a and n are constants. She has a table of values of P and t. Explain how she could use a graph to test her idea and to estimate a and n

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Model answer.

Plot $\log_{10} P$ against $\log_{10} t$ B1

If the points lie close to a straight line, the model is suitable B1

The gradient of the line gives n B1

The intercept on the vertical axis is $\log_{10} a$, so $a = 10^{\text{intercept}}$ B1

Don't lose marks

- Giving $k = 4.3$ instead of $k = 10^{4.3}$: forgetting to undo the log.

- Undoing with the wrong base, e.g. using e^c when the axis is $\log_{10}$
- Reading the intercept from a vertical axis that is not drawn at 0 on the horizontal scale.
- Plotting $\log y$ against $\log x$ for an exponential model, or $\log y$ against x for a power law.
- Rounding the gradient or intercept before raising 10 (or e) to that power.

More tips

Memory tricks

- Count the logs: only y logged means exponential ($y = kb^x$); both logged means power law ($y = ax^n$).
- The intercept is always the log of the constant in front ($\log a$ or $\log k$), so it always needs undoing.
- For a power law the gradient is n itself; for an exponential the gradient is $\log b$, so it needs undoing too.
- Sense check: a line sloping down in an exponential model means $0 < b < 1$, which is decay.

Exam technique

- In 'show that' parts, show the log laws step by step:
  $\log(kb^t) = \log k + \log b^t = \log k + t \log b$
- Give the constants to the accuracy asked for, usually 3 significant figures.
- When asked to interpret, use the context and units, e.g. 'k is the value of the car when it was new'.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade D — **Take logs of $y = ax^n$**
$\log y = n \log x + \log a$: a straight line when $\log y$ is plotted against $\log x$

Grade D — **Take logs of $y = kb^x$**
$\log y = x \log b + \log k$: a straight line when $\log y$ is plotted against x

Grade C — **Choose the right axes for a straight line**
Power law: $\log y$ against $\log x$. Exponential: $\log y$ against x

Grade B — **Find a and n from a line**
The gradient is n; the intercept c is $\log a$, so $a = 10^c$ (or $a = e^c$ if the axes use ln).

Grade B — **Find k and b from a line**
The intercept c is $\log k$ and the gradient m is $\log b$, so $k = 10^c$ and $b = 10^m$

Grade A — **Interpret the constants and criticise the model**
E.g. k is the value when $x = 0$ and b is the multiplier per unit of x; predictions far outside the data are unreliable.

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