Solving exponential equations

P6.5 · Pure mathematics · revision notes

Solving equations where the unknown is in the power, such as $a^x = b$, by taking logarithms of both sides. Harder questions have different bases on each side or are quadratics in disguise, such as $9^x - 4(3^x) - 21 = 0$. These are common 3 to 6 mark questions.

Key facts

- $a^x = b \Rightarrow x = \log_a b = \dfrac{\ln b}{\ln a}$
- Change of base: $\log_a b = \dfrac{\log_c b}{\log_c a}$
- $a^{x+k} = a^k \times a^x$ and $a^{x-k} = \dfrac{a^x}{a^k}$
- $\ln(a^x) = x \ln a$
- $4^x = (2^x)^2$, $9^x = (3^x)^2$, $\mathrm{e}^{2x} = (\mathrm{e}^x)^2$
- $a^x > 0$ always, so a root $u \leq 0$ of the quadratic gives no solution

Notes

The basic method

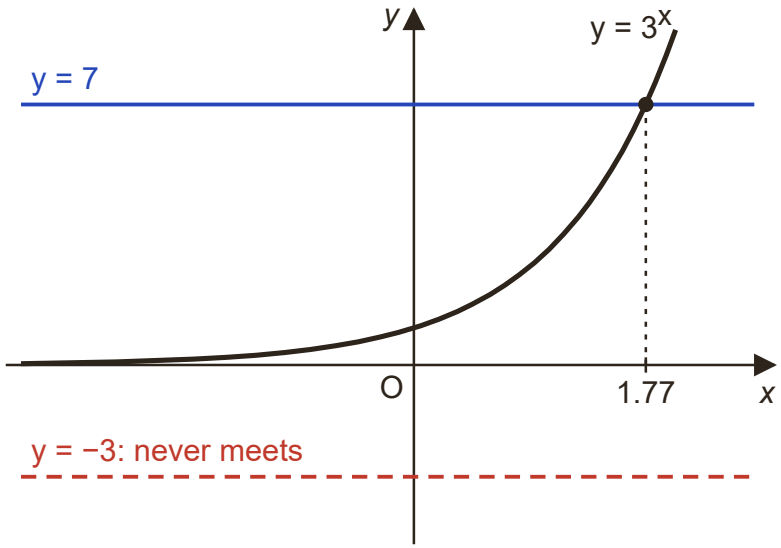
- To solve $a^x = b$ (where $b > 0$), take logs of both sides: $x \log a = \log b$, so $x = \dfrac{\log b}{\log a}$
- You can use ln or $\log_{10}$, as long as you use the same on both sides. Use ln whenever e is involved.
- The answer can also be written $x = \log_a b$. For example, $5^x = 40$ gives $x = \log_5 40 = \dfrac{\ln 40}{\ln 5} = 2.29$ (3 s.f.).
- $a^x = b$ has no solution if $b \leq 0$, because a^x is always positive.

Harder linear equations

- For $a^{px+q} = b$, the whole bracket comes down: $(px + q) \ln a = \ln b$
- With different bases, e.g. $3^{2x} = 7^{x+1}$: $2x \ln 3 = (x + 1) \ln 7$. Expand, collect the x terms and factorise: $x(2 \ln 3 - \ln 7) = \ln 7$, so $x = \dfrac{\ln 7}{2 \ln 3 - \ln 7}$
- Keep $\ln 3$, $\ln 7$ and so on exact until the final line, to avoid rounding errors.

Disguised quadratics

- Write each term using one exponential: $4^x = (2^x)^2$, $2^{x+1} = 2 \times 2^x$, $3^{x-1} = \dfrac{3^x}{3}$, $e^{2x} = (e^x)^2$
- Substitute $u = 2^x$ (or $u = e^x$), solve the quadratic in u, then solve $2^x = u$ for each root.
- Reject any root with $u \leq 0$, because $2^x > 0$ for all x. Write this reason in your answer.



$3^x = 7$ has one solution, $x = \dfrac{\ln 7}{\ln 3} = 1.77$; $3^x = -3$ has none, because $3^x > 0$

- If there is an e^{-x} term, multiply every term by e^x first ($e^x \times e^{-x} = 1$).

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, dM = depends on the earlier method mark.

Solve $a^{px+q} = b$ to a given accuracy — 2 to 3 marks — Grade D

1. Take logs of both sides and bring the power down as a bracket.
2. Rearrange for x, keeping the logs exact until the end.
3. Round to the accuracy asked for (often 3 significant figures).

Example. Solve $3^{2x-1} = 20$, giving your answer to 3 significant figures.

Model answer.

$(2x - 1)\ln 3 = \ln 20$ M1

$2x - 1 = \dfrac{\ln 20}{\ln 3} = 2.7268\ldots$, so $x = \dfrac{1}{2}\left(\dfrac{\ln 20}{\ln 3} + 1\right)$ dM1

$x = 1.86$ A1

Solve an equation with different bases

3 to 4 marks | **Grade B**

1. Take logs of both sides, using brackets for powers with more than one term.
2. Expand the brackets and collect all the x terms on one side.
3. Factorise out x and divide. Simplify using the log laws if an exact form is asked for.

Example. Solve $2^{x+3} = 5^x$, giving your answer
(a) in the form $x = \dfrac{\ln a}{\ln b}$, where a and b are rational numbers,
(b) to 3 significant figures.

Model answer.

(a) $(x+3)\ln 2 = x\ln 5$ M1

$3\ln 2 = x(\ln 5 - \ln 2)$ M1

$x = \dfrac{\ln 8}{\ln 2.5}$ A1

(b) $x = 2.27$ A1

Solve a disguised quadratic

4 to 6 marks | **Grade A**

1. Rewrite every term in terms of one exponential, e.g. $9^x = (3^x)^2$
2. Let $u = 3^x$ and solve the quadratic in u
3. Reject any value of u that is zero or negative, giving the reason.
4. Solve $3^x = u$ for the remaining value(s).

Example. Solve $9^x - 4(3^x) - 21 = 0$, giving your answer to 3 significant figures.

Model answer.

$9^x = (3^x)^2$, so with $u = 3^x$: $u^2 - 4u - 21 = 0$ B1

$(u-7)(u+3) = 0$, so $u = 7$ or $u = -3$ M1 A1

$3^x = -3$ has no solution because $3^x > 0$; $3^x = 7 \Rightarrow x = \dfrac{\ln 7}{\ln 3}$ M1

$x = 1.77$ A1

Solve an equation in e^x and e^{-x}

4 marks | Grade A

1. Multiply every term by e^x, remembering $e^x \times e^{-x} = 1$
2. Rearrange into a quadratic in e^x and factorise.
3. Reject a negative value of e^x, then take ln.

Example. Solve $e^x - 6e^{-x} = 1$, giving your answer in an exact form.

Model answer.

$e^{2x} - 6 = e^x$ M1, so $e^{2x} - e^x - 6 = 0$ A1

$(e^x - 3)(e^x + 2) = 0$ M1

$e^x = -2$ is impossible, so $e^x = 3$ and $x = \ln 3$ A1

Don't lose marks

- Losing the bracket: writing $2x - 1 \ln 3$ instead of $(2x - 1)\ln 3$
- Taking logs of each term separately: $\ln(4^x - 3 \times 2^x)$ is not $\ln 4^x - \ln(3 \times 2^x)$
- Writing 2^{x+1} as $2^x + 2$ instead of 2×2^x
- Not rejecting the negative root of the quadratic, or 'solving' $3^x = -3$ to get $x = -1$
- Rounding $\dfrac{\ln 20}{\ln 3}$ early, so the final answer is wrong in the last figure.

More tips

Memory tricks

- Spot the square: 4^x, 9^x, 25^x and e^{2x} all signal a quadratic in disguise.
- Estimate first: $2^4 = 16$ and $2^5 = 32$, so the solution of $2^x = 20$ is between 4 and 5.
- Check by substituting your answer back: $3^{2\times 1.8634 - 1}$ should give about 20.
- Put brackets round powers with two terms before you bring them down: $\ln(3^{2x-1}) = (2x - 1)\ln 3$

Exam technique

- Read the required form: exact, 'in the form $\dfrac{\ln a}{\ln b}$', or a number of significant figures or decimal places.
- If the question says 'show all stages of your working', write the log step; a calculator's equation solver alone gets no marks.
- When you reject a root, say why in words, e.g. '3^x cannot be negative'.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E **Solve $a^x = b$ by taking logarithms**

$x \log a = \log b$, so $x = \dfrac{\log b}{\log a}$; e.g. $5^x = 40$ gives $x = 2.29$ (3 s.f.).

Grade D **Solve $e^{kx} = b$ using ln**

E.g. $e^{0.4x} = 9$ gives $x = \dfrac{\ln 9}{0.4} = 5.49$ (3 s.f.).

Grade C **Solve $a^{px+q} = b$**

Bring the whole bracket down: $(px + q)\ln a = \ln b$, then rearrange.

Grade B **Solve equations with different bases**

E.g. $2^{x+3} = 5^x$: take logs, expand, collect the x terms and factorise.

Grade A **Solve disguised quadratics in a^x**

Write $9^x = (3^x)^2$, substitute $u = 3^x$, solve, then reject any $u \leq 0$

Grade A **Solve equations with e^x and e^{-x}**

Multiply every term by e^x to get a quadratic in e^x

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