Laws of logarithms

P6.4 · Pure mathematics · revision notes

The three laws of logarithms let you combine, split and simplify logs, and solve equations that contain them. Expect 'write as a single logarithm', 'express in terms of p and q' and 'solve' questions, usually worth 3 to 6 marks and often with the instruction to show all stages of your working.

Key facts

- $\log_a x + \log_a y = \log_a(xy)$
- $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$
- $k\log_a x = \log_a(x^k)$
- $\log_a \frac{1}{x} = -\log_a x$
- $\log_a \sqrt{x} = \frac{1}{2}\log_a x$
- $\log_a a = 1$, $\log_a 1 = 0$, $\log_a(a^k) = k$

Notes

The three laws

- For $x, y > 0$: $\log_a x + \log_a y = \log_a(xy)$

  $$\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$$

- $k\log_a x = \log_a(x^k)$ for any number k. In particular $\log_a \frac{1}{x} = -\log_a x$ ($k = -1$) and $\log_a \frac{1}{\sqrt{x}} = -\frac{1}{2}\log_a x$ ($k = -\frac{1}{2}$).
- These come from the laws of indices: multiplying powers of the same base adds the indices, so multiplying numbers adds their logs.
- Useful facts: $\log_a a = 1$, $\log_a 1 = 0$ and $\log_a(a^k) = k$

Writing as a single logarithm

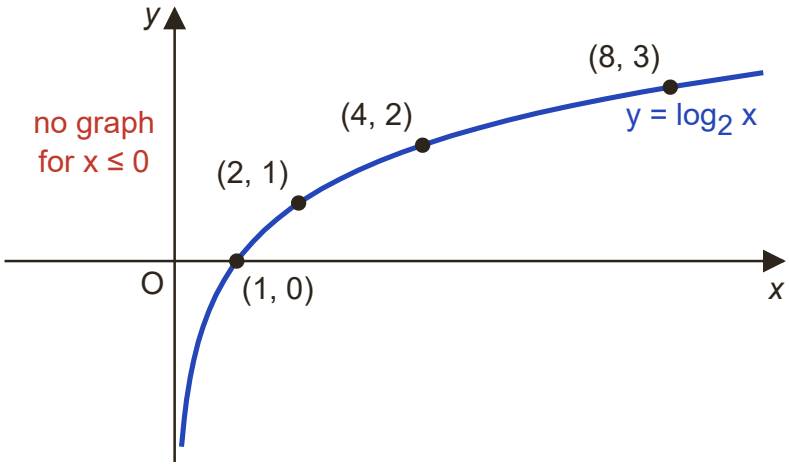
- Use the power law first (move any number in front inside as a power), then combine with the addition and subtraction laws.
- Example:

$$2\log x - \frac{1}{2}\log y + \log 5 = \log x^2 - \log \sqrt{y} + \log 5$$
$$= \log \frac{5x^2}{\sqrt{y}}$$

- Write plain numbers as logs to the same base: $3 = \log_2 8$ and $1 = \log_a a$

Solving log equations

- Get a single log on each side. Then $\log_a f(x) = \log_a g(x)$ gives $f(x) = g(x)$, and $\log_a f(x) = c$ gives $f(x) = a^c$
- Check every answer in the **original** equation: each log must have a positive argument. Reject any answer that fails and say why.



A log only exists for a positive input, so $\log_2(-2)$ is not defined and $x = -2$ must be rejected.

- The laws work the same way for ln: $\ln x + \ln 3 = \ln 3x$

Things that are not laws

- $\log(x + y) \neq \log x + \log y$, $(\log x)^2 \neq 2\log x$ and $\dfrac{\log x}{\log y} \neq \log \dfrac{x}{y}$

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Write as a single logarithm

2 to 3 marks | **Grade C**

1. Use the power law to move any coefficients inside as powers.
2. Write any plain number as a log to the same base.
3. Combine: plus means multiply, minus means divide.

Example. Express $2\log_3 x - \log_3(x+4) + 2$ as a single logarithm to base 3.

Model answer.

$2\log_3 x = \log_3 x^2$ M1

$2 = \log_3 9$ B1

$\log_3 \dfrac{9x^2}{x+4}$ A1

Express in terms of given logarithms

3 to 4 marks | **Grade C**

1. Write the number as a product or quotient of the given numbers (and the base).
2. Split with the addition and subtraction laws, then use the power law.
3. Replace each log with the given letter, remembering $\log_a a = 1$

Example. Given that $\log_a 2 = p$ and $\log_a 3 = q$, express in terms of p and q:
(a) $\log_a 12$
(b) $\log_a \dfrac{9a}{2}$

Model answer.

(a) $\log_a 12 = \log_a(2^2 \times 3)$
$= 2\log_a 2 + \log_a 3$ M1 $= 2p + q$ A1

(b) $\log_a 9 + \log_a a - \log_a 2$ M1 $= 2q + 1 - p$ A1

Solve a logarithmic equation

4 to 5 marks | **Grade B**

1. Combine the logs into a single log.
2. Change to index form (or remove logs from both sides).
3. Solve the resulting equation.
4. Check each solution in the original equation and reject any that give the log of a negative number or zero.

Example. Solve $\log_2 x + \log_2(x-2) = 3$

Model answer.

$\log_2 x(x-2) = 3$ M1

$x(x-2) = 2^3 = 8$ M1

$x^2 - 2x - 8 = 0 \Rightarrow (x-4)(x+2) = 0$ A1

$x = 4$ ($x = -2$ is rejected because $\log_2(-2)$ is not defined) A1

Solve a log equation that leads to a quadratic

5 marks | **Grade A**

1. Use the power law, then combine into a single log on each side.
2. Remove the logs and rearrange into a quadratic.
3. Solve, giving exact answers.
4. Check each root in the original equation.

Example. Solve $2\ln(x+1) - \ln x = \ln 6$, giving your answers in an exact form.

Model answer.

$\ln \dfrac{(x+1)^2}{x} = \ln 6$ M1

$(x+1)^2 = 6x$ M1

$x^2 - 4x + 1 = 0$ A1

$x = \dfrac{4 \pm \sqrt{12}}{2}$ M1 $= 2 \pm \sqrt{3}$ A1

Both values are positive, so every log in the original equation is defined and both answers are valid.

Don't lose marks

- Using $\log(x+y) = \log x + \log y$, which is not a law.
- Not rejecting a solution that gives the log of a negative number or zero, or rejecting one without giving a reason.

- Stopping at $\log_3 \dfrac{x^2}{x+4} + 2$: the 2 must also be written as a log to give a single logarithm.
- Thinking $\dfrac{\log 8}{\log 2} = \log 4$. In fact $\dfrac{\log 8}{\log 2} = 3$

More tips

Memory tricks

- Power law first, then combine.
- Plus becomes times, minus becomes divide, a number in front becomes a power.
- Check a 'single log' answer on your calculator: pick a value such as $x = 2$ and evaluate both the original expression and your answer.

Exam technique

- In 'show that' questions, use one law per line so every step can be seen.
- When you reject a root, write the reason, e.g. '$x = -2$ is rejected because $\log_2(-2)$ is undefined'.
- Many log questions say 'show all stages of your working': a calculator's equation solver alone scores nothing.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Use the addition and subtraction laws**

$\log_a x + \log_a y = \log_a xy$ and $\log_a x - \log_a y = \log_a \dfrac{x}{y}$

Grade D — **Use the power law, including negative powers**

$3\log_a x = \log_a x^3$, $-\log_a x = \log_a \dfrac{1}{x}$ and $-\dfrac{1}{2}\log_a x = \log_a \dfrac{1}{\sqrt{x}}$

Grade C — **Write an expression as a single logarithm**

Use the power law first, then combine, writing plain numbers as logs, e.g. $2 = \log_3 9$

Grade C — **Express logarithms in terms of given logs**

Given $\log_a 2 = p$ and $\log_a 3 = q$, write $\log_a 12 = 2p + q$

Grade B — **Solve equations involving logarithms**

Combine into one log, then change to index form, e.g. $\log_2 x(x-2) = 3 \Rightarrow x(x-2) = 8$

Grade A — **Solve log equations leading to quadratics**

Form and solve a quadratic, then check every root in the original equation, rejecting any that make a log's argument zero or negative.

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