Logarithms

P6.3 · Pure mathematics · revision notes

A logarithm is the inverse of an exponential: $\log_a b$ is the power you raise a to in order to get b. You need to switch between index form and log form, use the natural logarithm $\ln x$ as the inverse of e^x, and sketch $y = \ln x$. These skills are used in almost every later exponential and log question.

Key facts

- $\log_a b = c \Leftrightarrow a^c = b$
- $\log_a x$ only exists for $x > 0$
- $\ln(\mathrm{e}^x) = x$; $\mathrm{e}^{\ln x} = x$ for $x > 0$
- $y = \ln x$ is $y = \mathrm{e}^x$ reflected in $y = x$
- $\log_a 1 = 0$ and $\log_a a = 1$
- $\ln x = \log_{\mathrm{e}} x$
- $y = \ln x$: through $(1, 0)$, asymptote $x = 0$, domain $x > 0$

Notes

What a logarithm is

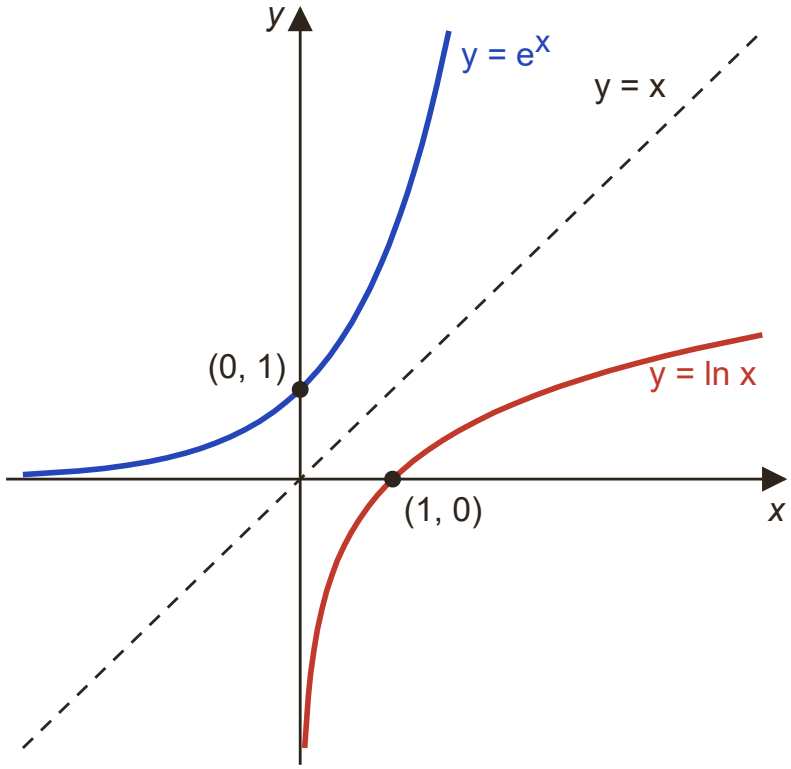
- $\log_a b = c$ means $a^c = b$. The logarithm is the power you raise the base a to, to get b
- The base a is positive and not equal to 1, and you can only take the log of a positive number: $\log_a x$ is defined only for $x > 0$
- $\log_a 1 = 0$ because $a^0 = 1$, and $\log_a a = 1$ because $a^1 = a$
- Examples: $\log_3 81 = 4$ because $3^4 = 81$; $\log_2 \frac{1}{8} = -3$ because $2^{-3} = \frac{1}{8}$; $\log_{25} 5 = \frac{1}{2}$ because $25^{1/2} = 5$
- $\log_a x$ is the inverse of a^x: $\log_a(a^x) = x$ and $a^{\log_a x} = x$

Natural logarithms

- $\ln x$ means $\log_{\mathrm{e}} x$, the logarithm to base e. It is the inverse of e^x
- $\ln(\mathrm{e}^x) = x$ for all x, and $\mathrm{e}^{\ln x} = x$ for $x > 0$. Also $\ln 1 = 0$ and $\ln \mathrm{e} = 1$
- To solve $\mathrm{e}^{(\ldots)} = k$, take ln of both sides. To solve $\ln(\ldots) = k$, write $(\ldots) = \mathrm{e}^k$

The graph of $y = \ln x$

- $y = \ln x$ is the reflection of $y = \mathrm{e}^x$ in the line $y = x$, so the point $(0, 1)$ becomes $(1, 0)$ and the asymptote $y = 0$ becomes $x = 0$



$y = \ln x$ is $y = \mathrm{e}^x$ reflected in $y = x$: each asymptote becomes the other axis.

- Domain $x > 0$; range all real numbers. The curve always increases, but more and more slowly.
- $y = \log_a x$ (for $a > 1$) has the same shape and also passes through $(1, 0)$
- Transformations move the asymptote: $y = \ln(x - 2)$ has asymptote $x = 2$ and crosses the x-axis where $x - 2 = 1$, at $(3, 0)$

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Write in log form or evaluate without a calculator

1 to 4 marks | Grade E

1. Ask yourself: 'the base to what power gives this number?'
2. Write the number as a power of the base, e.g. $8 = 4^{3/2}$ or $\frac{1}{27} = 3^{-3}$
3. The power is the value of the logarithm.

Example. (a) Write $2^5 = 32$ using logarithms.
(b) Find the value of $\log_4 8$.
(c) Find the value of $\log_3 \frac{1}{27}$

Model answer.

(a) $\log_2 32 = 5$ B1

(b) $8 = 2^3$
$= (4^{1/2})^3$
$= 4^{3/2}$ M1, so $\log_4 8 = \frac{3}{2}$ A1

(c) $\frac{1}{27} = 3^{-3}$, so $\log_3 \frac{1}{27} = -3$ B1

Solve equations with e and ln

2 to 4 marks | Grade C

1. Make the exponential or the log the subject of the equation.
2. Apply the inverse: take ln of both sides of $e^{(\ldots)} = k$, or write $(\ldots) = e^k$ if $\ln(\ldots) = k$
3. Rearrange for the unknown and leave the answer exact unless told otherwise.

Example. Solve, giving each answer in an exact form:
(a) $e^{3x+1} = 10$
(b) $\ln(2y - 1) = 4$

Model answer.

(a) $3x + 1 = \ln 10$ M1, so $x = \frac{\ln 10 - 1}{3}$ A1

(b) $2y - 1 = e^4$ M1, so $y = \frac{e^4 + 1}{2}$ A1

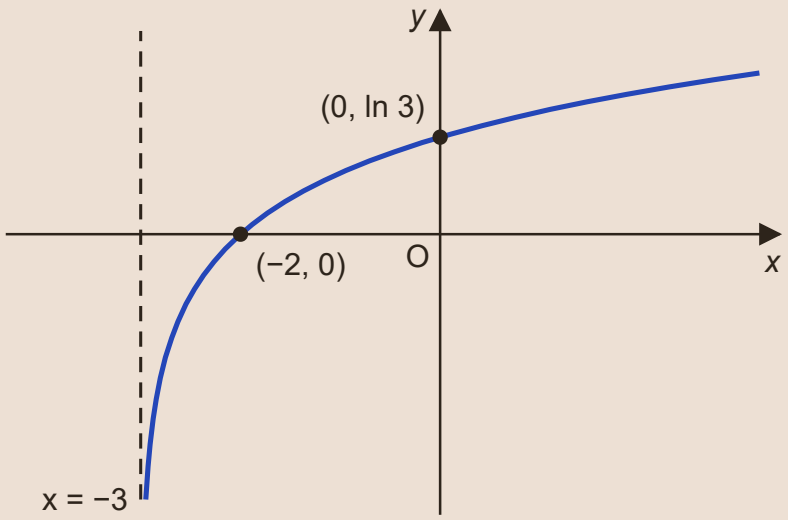
Sketch a transformed log graph

3 to 4 marks **Grade B**

1. Find the vertical asymptote: where the expression inside the ln equals 0.
2. Find the x-intercept: where the expression inside the ln equals 1.
3. Find the y-intercept by putting $x = 0$ (if the function is defined there).
4. Draw the shape and label the intercepts and the asymptote.

Example. Sketch the graph of $y = \ln(x + 3)$, showing the coordinates of the points where it meets the axes and the equation of its asymptote.

Model answer.



Asymptote $x = -3$ B1

$y = 0$: $x + 3 = 1$, so $(-2, 0)$ B1

$x = 0$: $y = \ln 3$, so $(0, \ln 3)$ B1

Correct shape: an increasing curve to the right of $x = -3$, with $y \to -\infty$ as x gets closer to -3 B1

Find the inverse of an exponential function

3 to 4 marks | **Grade A**

1. Write $y = f(x)$ and make x the subject, using ln to undo e
2. Write $f^{-1}(x)$ in terms of x
3. The domain of f^{-1} is the range of f

Example. The function f is defined by $f(x) = 3 + \mathrm{e}^{2x}$, $x \in \mathbb{R}$. Find $f^{-1}(x)$ and state its domain.

Model answer.

$y = 3 + \mathrm{e}^{2x} \Rightarrow \mathrm{e}^{2x} = y - 3$ M1

$2x = \ln(y-3)$, so $x = \frac{1}{2}\ln(y-3)$ A1

$f^{-1}(x) = \frac{1}{2}\ln(x-3)$ A1

Domain $x > 3$ (because the range of f is $f(x) > 3$) B1

Don't lose marks

- Taking the log of zero or a negative number, or not rejecting a solution that makes the expression inside a log zero or negative.
- Writing $\ln(x+3) = \ln x + \ln 3$. There is no law like this.
- Mixing up the asymptotes: $y = \ln x$ has a vertical asymptote $x = 0$; $y = \mathrm{e}^x$ has a horizontal asymptote $y = 0$
- Writing $\mathrm{e}^{\ln x + 2} = x + 2$. In fact $\mathrm{e}^{\ln x + 2} = \mathrm{e}^{\ln x} \times \mathrm{e}^2 = \mathrm{e}^2 x$

More tips

Memory tricks

- A log asks a question: $\log_2 32$ asks '2 to what power gives 32?'
- ln and e cancel each other out because they are inverses: $\mathrm{e}^{\ln 5} = 5$ and $\ln \mathrm{e}^7 = 7$
- For a base bigger than 1, the log of a number between 0 and 1 is negative, and the log of 1 is 0.
- The graph of $y = \ln x$ is $y = \mathrm{e}^x$ with the coordinates swapped: $(0, 1)$ becomes $(1, 0)$, and the asymptote $y = 0$ becomes $x = 0$

Exam technique

- Give exact answers in terms of ln and e when asked; only use decimals at the end if the question wants them.
- When you find an inverse, always state its domain: it is the range of the original function.
- Label asymptotes with their equations on every sketch.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Convert between index form and log form**

$\log_a b = c$ means exactly the same as $a^c = b$

Grade E — **Evaluate simple logarithms without a calculator**

For example $\log_2 32 = 5$, $\log_5 1 = 0$ and $\log_9 3 = \frac{1}{2}$

Grade D — **Use ln and e as inverse functions**

$\ln(\mathrm{e}^x) = x$ for all x, and $\mathrm{e}^{\ln x} = x$ for $x > 0$

Grade C — **Solve equations using ln and e**

Take ln of both sides of $\mathrm{e}^{3x+1} = 10$, or write $2y - 1 = \mathrm{e}^4$ when $\ln(2y - 1) = 4$

Grade C — **Sketch $y = \ln x$**

Through $(1, 0)$, increasing, defined only for $x > 0$, with the y-axis as a vertical asymptote.

Grade B — **Sketch transformed log graphs**

E.g. $y = \ln(x + 3)$ has asymptote $x = -3$ and meets the axes at $(-2, 0)$ and $(0, \ln 3)$

Grade A — **Find inverses of exponential and log functions**

E.g. for $f(x) = 3 + \mathrm{e}^{2x}$, $f^{-1}(x) = \frac{1}{2}\ln(x - 3)$ with domain $x > 3$

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