The function e^x and its gradient

P6.2 · Pure mathematics · revision notes

The special property of e^x: the gradient of $y = e^{kx}$ is ke^{kx}, so the gradient is proportional to the value of the function. This is why exponential models fit anything that grows or decays at a rate proportional to its size. It comes up in gradient questions and when you are asked to explain a model.

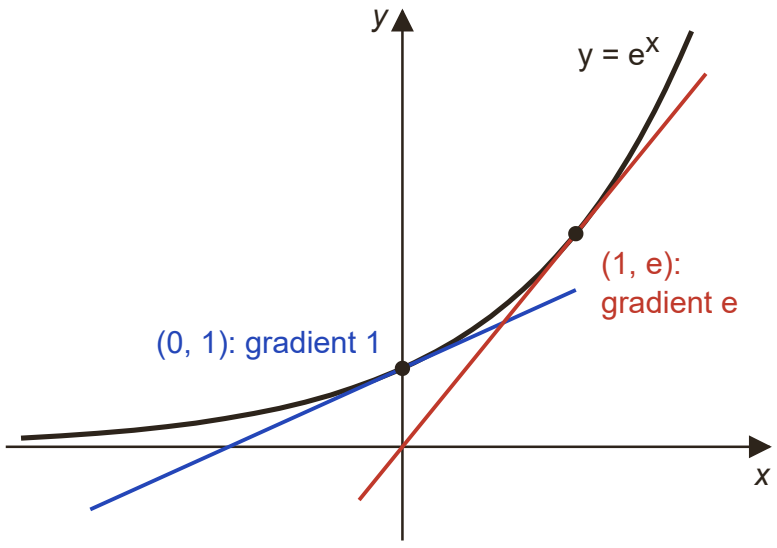
Key facts

- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(e^{kx}) = ke^{kx}$
- $\frac{d}{dx}(Ae^{kx}) = Ake^{kx}$
- If $y = Ae^{kx}$, then $\frac{dy}{dx} = ky$
- The gradient of $y = e^x$ at $(0, 1)$ is 1
- Rate of change proportional to the amount ⇔ exponential model

Notes

The gradient of e^x

- For $y = e^x$, the gradient at every point equals the y-coordinate: $\frac{dy}{dx} = e^x$
- So at $(0, 1)$ the gradient is 1, and at $(2, e^2)$ the gradient is e^2



At every point of $y = e^x$ the gradient equals the y-coordinate.

- e is the only base for which this is true. For any other $y = a^x$ the gradient is a constant multiple of a^x (the constant is $\ln a$).

Differentiating e^{kx}

- $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{kx}\right) = k\mathrm{e}^{kx}$ for any constant k: the power stays the same and k comes down in front.
- Constant multiples stay: $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(A\mathrm{e}^{kx}\right) = Ak\mathrm{e}^{kx}$. For example, $y = 5\mathrm{e}^{-2x}$ gives $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -10\mathrm{e}^{-2x}$
- If $k > 0$ the gradient is positive and increases as x increases. If $k < 0$ the gradient is negative and gets closer to 0.
- Since $A\mathrm{e}^{kx} = y$, the result can be written $\dfrac{\mathrm{d}y}{\mathrm{d}x} = ky$: the gradient is proportional to y

Why exponential models are so useful

- Many quantities change at a rate proportional to their size: a population (more animals, more births), the number of undecayed radioactive nuclei, money earning continuous interest, the amount of a drug in the blood.
- If $\dfrac{\mathrm{d}N}{\mathrm{d}t} = kN$, then $N = N_0\mathrm{e}^{kt}$ fits, because differentiating $N_0\mathrm{e}^{kt}$ gives $kN_0\mathrm{e}^{kt} = kN$
- $k > 0$ gives growth that gets faster and faster; $k < 0$ gives decay that gets slower and slower.
- In an 'explain' question, say that the rate of change is proportional to the amount, and put it in context.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Find the exact gradient at a point 2 to 3 marks **Grade D**

1. Differentiate: multiply by the number in front of x in the power.
2. Substitute the x-value into $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, not into y
3. Leave the answer in terms of e if it asks for an exact value.

Example. The curve C has equation $y = 4\mathrm{e}^{0.5x}$. Find the exact gradient of C at the point where $x = 2$

Model answer.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4 \times 0.5\mathrm{e}^{0.5x}$ M1 $= 2\mathrm{e}^{0.5x}$ A1

At $x = 2$: gradient $= 2\mathrm{e}^{1} = 2\mathrm{e}$ A1

Find the point where the gradient takes a given value

3 to 4 marks | Grade B

1. Differentiate the whole expression.
2. Set $\frac{\mathrm{d}y}{\mathrm{d}x}$ equal to the given gradient.
3. Make the exponential the subject, then take ln of both sides.

Example. The curve C has equation $y = 3\mathrm{e}^{2x} - 5x$. The point P on C has gradient 7. Find the exact x-coordinate of P

Model answer.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 6\mathrm{e}^{2x} - 5$ M1 A1

$6\mathrm{e}^{2x} - 5 = 7 \Rightarrow \mathrm{e}^{2x} = 2$ M1

$x = \frac{1}{2}\ln 2$ A1

Show a rate is proportional to the amount, and explain

2 to 3 marks | Grade A

1. Differentiate the model.
2. Rewrite the derivative in terms of the original variable, e.g. $\frac{\mathrm{d}N}{\mathrm{d}t} = 0.3N$
3. Interpret in context: the rate of increase is proportional to the current amount.

Example. The number of bacteria, N, in a culture t hours after the start of an experiment is modelled by $N = 200\mathrm{e}^{0.3t}$.

(a) Show that $\frac{\mathrm{d}N}{\mathrm{d}t} = 0.3N$.

(b) Explain what this tells you about how the number of bacteria grows.

Model answer.

(a) $\frac{\mathrm{d}N}{\mathrm{d}t} = 200 \times 0.3\mathrm{e}^{0.3t}$ M1 $= 0.3 \times 200\mathrm{e}^{0.3t} = 0.3N$ A1

(b) The rate at which the number of bacteria increases is proportional to the number of bacteria present, so the more bacteria there are, the faster the number grows B1

Don't lose marks

- Using the power rule on e^x, e.g. writing $x\mathrm{e}^{x-1}$. The power rule is only for x^n
- Leaving out the k: the derivative of e^{3x} is $3\mathrm{e}^{3x}$, not e^{3x}
- Sign slips with negative powers: the derivative of e^{-x} is $-\mathrm{e}^{-x}$
- Substituting into y instead of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when asked for a gradient.

✗ Giving 5.44 when the question says 'exact': the answer is $2\mathrm{e}$

More tips

Memory tricks

- Bring the k down in front: $\mathrm{e}^{kx} \to k\mathrm{e}^{kx}$. The power itself never changes.
- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = ky$ is the fingerprint of an exponential: if a rate is proportional to the amount, think e
- Quick check: the tangent to $y = \mathrm{e}^x$ at $(0, 1)$ is $y = x + 1$, with gradient 1.

Exam technique

- 'Exact' means keep e and ln in the answer; do not round.
- In 'explain' questions use the phrase 'the rate of change (or rate of increase) is proportional to' and name the quantity in the context.
- 'Show that' answers need the step where you replace $200\mathrm{e}^{0.3t}$ by N; do not jump straight to the result.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **State the gradient function of e^x**

If $y = \mathrm{e}^x$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$: the gradient at every point equals the y-coordinate.

Grade D — **Differentiate e^{kx} and $A\mathrm{e}^{kx}$**

Multiply by k: the derivative of $5\mathrm{e}^{-2x}$ is $-10\mathrm{e}^{-2x}$

Grade C — **Find the gradient at a given point**

Differentiate first, then substitute, giving exact answers such as $2\mathrm{e}$

Grade B — **Find where the gradient has a given value**

Solve an equation such as $6\mathrm{e}^{2x} - 5 = 7$ using ln.

Grade A — **Explain why exponential models suit many situations**

For $y = A\mathrm{e}^{kx}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = ky$: the rate of change is proportional to the amount present.

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