Exponential functions and graphs

P6.1 · Pure mathematics · revision notes

The graphs of $y = a^x$ (for $a > 0$) and $y = \mathrm{e}^x$: their shape, the point $(0, 1)$, the horizontal asymptote, and transformed versions such as $y = 5 - 2\mathrm{e}^{-x}$. It usually comes up as a 3 to 5 mark sketch, or as the first part of a modelling question.

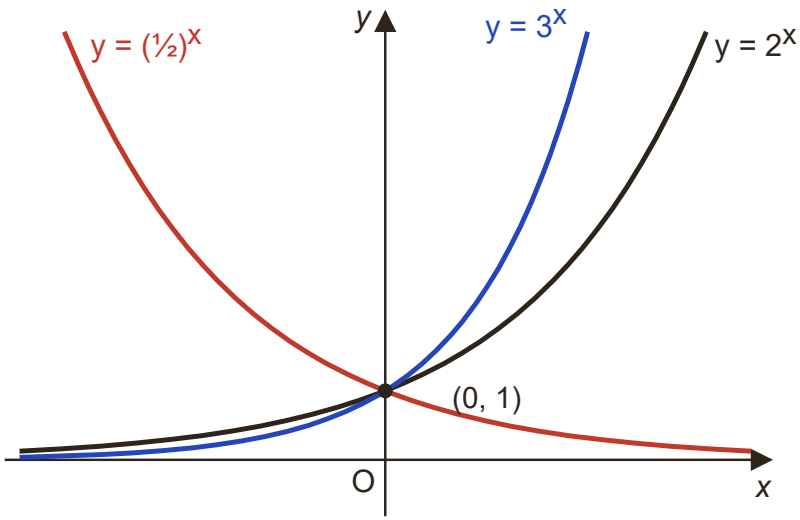
Key facts

- $a^0 = 1$, so $y = a^x$ passes through $(0, 1)$
- $a > 1$: growth; $0 < a < 1$: decay
- $\mathrm{e} \approx 2.718$ (irrational)
- $\mathrm{e}^x \to 0$ as $x \to -\infty$; $\mathrm{e}^{-x} \to 0$ as $x \to \infty$
- $a^x > 0$ for all x (when $a > 0$)
- $y = a^{-x}$ is $y = a^x$ reflected in the y-axis
- $y = A\mathrm{e}^{kx} + c$: y-intercept $(0, A + c)$, asymptote $y = c$

Notes

The graph of $y = a^x$

- For any constant $a > 0$, a^x is defined for every real x and is always positive, so the graph never touches or crosses the x-axis.
- Every graph $y = a^x$ passes through $(0, 1)$, because $a^0 = 1$
- If $a > 1$ the graph increases (exponential growth). As $x \to -\infty$, $y \to 0$, so the x-axis ($y = 0$) is a horizontal asymptote.
- If $0 < a < 1$ the graph decreases (exponential decay). Since $a^{-x} = \left(\frac{1}{a}\right)^x$, the graph of $y = a^{-x}$ is the reflection of $y = a^x$ in the y-axis.
- For $a > 1$, a bigger base gives a steeper curve for $x > 0$ and a curve closer to the x-axis for $x < 0$



All pass through $(0, 1)$ and have the x-axis as an asymptote. The bigger base is steeper for $x > 0$; $y = (\frac{1}{2})^x$ is the reflection of $y = 2^x$ in the y-axis.

The exponential function e^x

- $e = 2.71828\ldots$ is an irrational number, and $y = e^x$ is called **the** exponential function.
- Its graph lies between $y = 2^x$ and $y = 3^x$. Its domain is all real numbers and its range is $y > 0$
- $y = e^{kx}$ shows growth if $k > 0$ and decay if $k < 0$. The bigger $|k|$ is, the steeper the curve.
- $y = e^{-x}$ is the reflection of $y = e^x$ in the y-axis: it decreases and passes through $(0, 1)$

Transformed exponential graphs

- Use the graph transformations you already know: $y = \mathrm{e}^x + c$ is a vertical translation by c, which moves the asymptote from $y = 0$ to $y = c$
- For $y = A\mathrm{e}^{kx} + c$: the y-intercept is $(0, A + c)$ and the asymptote is $y = c$. If $A > 0$ the curve lies above its asymptote; if $A < 0$ it lies below.
- To find where the curve crosses the x-axis, set $y = 0$ and solve using ln. If the equation has no solution (for example $\mathrm{e}^x = -2$), the curve does not cross the x-axis.
- On a sketch, label the asymptote with its equation and give exact coordinates of the intercepts.

How to answer each type of question

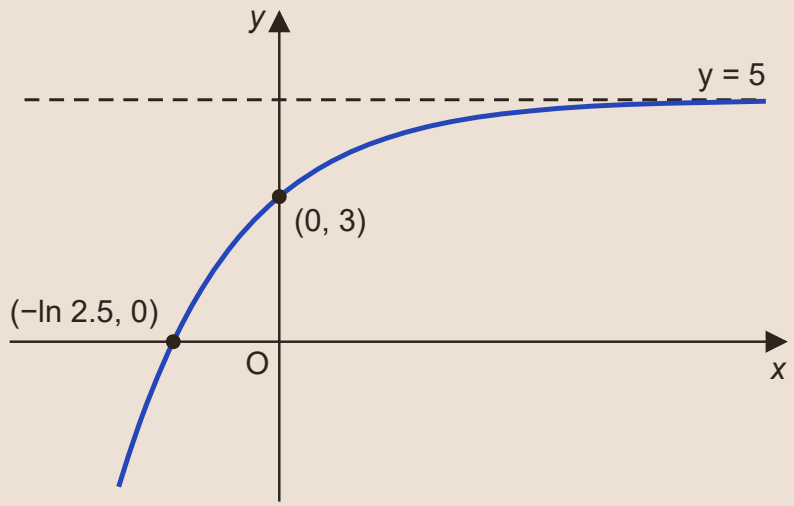
M = method mark, A = answer mark, B = a mark on its own.

Sketch a transformed exponential graph 3 to 5 marks **Grade C**

1. Find the y-intercept by putting $x = 0$
2. Find the asymptote: the value y approaches as the exponential term tends to 0.
3. Find any x-intercept by setting $y = 0$ and using ln.
4. Draw the correct shape, getting closer and closer to the asymptote without touching it, and label everything.

Example. Sketch the curve with equation $y = 5 - 2\mathrm{e}^{-x}$. On your sketch, show the exact coordinates of the points where the curve crosses the coordinate axes, and state the equation of the asymptote.

Model answer.



Shape: an increasing curve that lies below its asymptote and levels off as x increases	B1
When $x = 0$: $y = 5 - 2 = 3$, so $(0, 3)$	B1
When $y = 0$: $\mathrm{e}^{-x} = 2.5$ M1, so $x = -\ln 2.5$, giving $(-\ln 2.5, 0)$ (or $(\ln 0.4, 0)$)	A1
As $x \to \infty$, $\mathrm{e}^{-x} \to 0$, so the asymptote is $y = 5$	B1

State the intercepts, asymptote and range 3 to 4 marks **Grade D**

1. Put $x = 0$ for the y-intercept.
2. Put $y = 0$ and solve for the x-intercept, if there is one.
3. The asymptote is the constant the curve approaches; the range is every y-value on one side of it, not including it.

Example. The curve C has equation $y = 3^x - 9$.
(a) Find the coordinates of the points where C meets the coordinate axes.
(b) State the equation of the asymptote to C and the range of values of y

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Model answer.

(a) $x = 0$: $y = 1 - 9 = -8$, so $(0, -8)$ B1

$y = 0$: $3^x = 9$, so $x = 2$, giving $(2, 0)$ B1

(b) Asymptote $y = -9$ B1

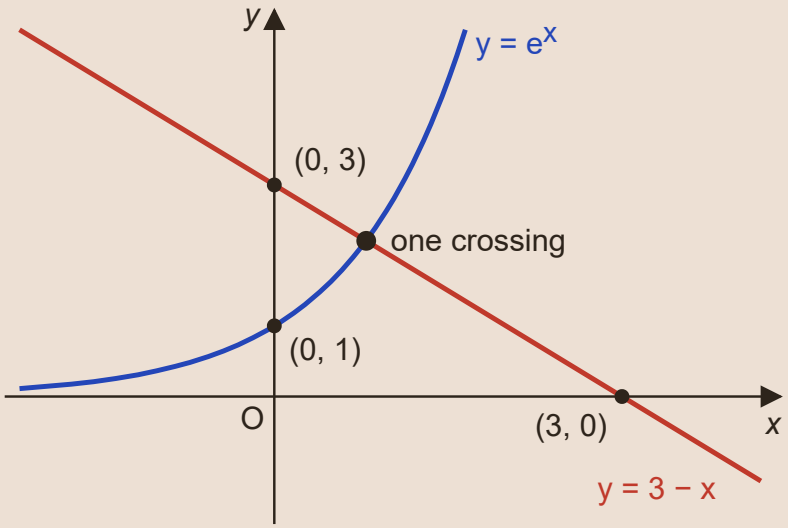
Range $y > -9$ B1

Use graphs to explain the number of solutions 3 to 4 marks **Grade A**

1. Rearrange the equation so each side is a graph you can sketch, e.g. $\mathbf{e}^{x} = 3 - x$
2. Sketch both graphs on the same axes with their intercepts.
3. Count the intersections and give a reason, e.g. one graph is increasing and the other is decreasing.

Example. (a) On the same axes, sketch the graphs of $y = \mathbf{e}^{x}$ and $y = 3 - x$.
(b) Hence explain why the equation $\mathbf{e}^{x} + x - 3 = 0$ has exactly one real root.

Model answer.



(a) Exponential curve through $(0, 1)$ with asymptote $y = 0$ B1; straight line through $(0, 3)$ and $(3, 0)$ B1

(b) The equation can be written $\mathbf{e}^{x} = 3 - x$ M1. $y = \mathbf{e}^{x}$ is increasing and $y = 3 - x$ is decreasing, so they can meet at most once, and they do meet (at $x = 0$ the line is above the curve; at $x = 3$ it is below). So there is exactly one real root A1

Don't lose marks

- Letting an exponential curve touch or cross its asymptote, or bend away from it.
- Leaving the equation of the asymptote off the sketch.
- Giving an intercept only as a rounded decimal when exact coordinates were asked for, e.g. writing -0.916 instead of $-\ln 2.5$
- Thinking $\mathbf{e}^{-x}$ can be negative. It is always positive; the minus sign in the power only reflects the graph.
- Writing the range as $y \geq c$: the curve never reaches its asymptote, so it is $y > c$ (or $y < c$).

More tips

Memory tricks

- Every basic exponential passes through $(0, 1)$: check your sketch goes through it before you apply any transformations.
- The asymptote comes from the constant term: in $y = Ae^{kx} + c$ it is $y = c$
- The sign of k gives the direction: e^{kx} rises from left to right if $k > 0$ and falls if $k < 0$
- Check on your calculator: substitute a large value such as $x = 20$ (or $x = -20$) to see the curve settle at its asymptote.

Exam technique

- 'Sketch' means show the shape and the key features (intercepts and asymptotes) clearly. You do not need to plot points, but the curve must not look like a straight line.
- Label every intercept with its coordinates and every asymptote with its equation.
- Keep intercepts exact (in ln form) unless the question asks for decimals.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Sketch $y = a^x$ for $a > 1$**

An increasing curve through $(0, 1)$ that gets closer and closer to the x-axis as $x \to -\infty$ but never touches it.

Grade D — **Sketch $y = a^x$ for $0 < a < 1$**

A decreasing curve through $(0, 1)$; it is the reflection of $y = \left(\frac{1}{a}\right)^x$ in the y-axis.

Grade C — **Know the graphs of $y = e^x$ and $y = e^{-x}$**

$e \approx 2.718$, so $y = e^x$ lies between $y = 2^x$ and $y = 3^x$; $y = e^{-x}$ is its reflection in the y-axis.

Grade B — **Sketch transformed exponential graphs**

For $y = Ae^{kx} + c$, show the y-intercept $(0, A + c)$ and the asymptote $y = c$

Grade B — **Find exact axis intercepts using logarithms**

Set $y = 0$ and solve with ln, e.g. $4 - e^{2x} = 0$ gives $x = \frac{1}{2}\ln 4 = \ln 2$

Grade A — **Use exponential graphs to count solutions**

Sketch both sides of an equation such as $e^x = 3 - x$ and explain why the graphs meet exactly once.

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