Proving trigonometric identities

P5.8 · Pure mathematics · revision notes

You prove trigonometric identities by turning one side into the other using identities you know. Questions say 'prove that' or 'show that', and often have a 'hence' part where you use the identity to solve an equation. Clear, step-by-step working is essential because the answer is given.

Key facts

- ≡ means true for all values of the variable (where defined)
- $1 + \tan^2 x \equiv \sec^2 x$ and $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$
- $1 + \cos 2x \equiv 2\cos^2 x$ and $1 - \cos 2x \equiv 2\sin^2 x$
- $\sec x \equiv \dfrac{1}{\cos x}$, $\operatorname{cosec} x \equiv \dfrac{1}{\sin x}$, $\cot x \equiv \dfrac{\cos x}{\sin x}$
- $\sin^2 x + \cos^2 x \equiv 1$
- $\sin 2x \equiv 2\sin x \cos x$
- $a^2 - b^2 \equiv (a - b)(a + b)$

Notes

Setting out a proof

- Start with one side, usually the more complicated one, and work towards the other side.
- Every line must follow from the one before, using a known identity or algebra. Write each step on a new line.
- Do not treat the identity as an equation: don't do the same thing to both sides, and don't cross-multiply.
- Finish with the other side, e.g. '= 2 sec x = RHS', or with a short concluding sentence.
- If it is hard to get from one side to the other, simplify each side separately until they meet.

Strategies that usually work

- Write sec, cosec, cot and tan in terms of sin and cos.
- Combine fractions into a single fraction with a common denominator.
- Look for $\sin^2 x + \cos^2 x = 1$ in disguise, e.g. $1 - \sin^2 x = \cos^2 x$ or $\sec^2 x - \tan^2 x = 1$
- Factorise, especially a difference of two squares: $\sec^4 x - \tan^4 x = (\sec^2 x - \tan^2 x)(\sec^2 x + \tan^2 x)$
- Change double angles into single angles, choosing the form of $\cos 2x$ that will cancel: $1 + \cos 2x = 2\cos^2 x$ and $1 - \cos 2x = 2\sin^2 x$
- Multiply the top and bottom of a fraction by the same expression, e.g. $1 + \sin x$, to create $1 - \sin^2 x = \cos^2 x$
- Keep the target in mind: if it contains only tan, you will probably need to divide sin by cos at some point.

'Hence' parts

- After proving an identity you may be asked to solve an equation 'hence'. Replace the complicated expression with the simpler side of the identity.
- Check that your solutions don't make any part of the original expression undefined, e.g. by giving a zero denominator.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, A1* = the solution must be fully correct, with every step shown.

Prove an identity with double angles, then use it ('hence') 5 marks **Grade B**

1. Replace $\sin 2\theta$ with $2\sin\theta\cos\theta$
2. Choose $\cos 2\theta = 2\cos^2\theta - 1$ so that the 1s cancel.
3. Cancel and conclude.
4. For 'hence', replace the left-hand side with tan θ and solve.

Example. (a) Prove that $\dfrac{\sin 2\theta}{1+\cos 2\theta} \equiv \tan\theta$.

(b) Hence solve $\dfrac{\sin 2\theta}{1+\cos 2\theta} = 3$ for $0° < \theta < 180°$, giving your answer to 1 decimal place.

Model answer.

(a) LHS $= \dfrac{2\sin\theta\cos\theta}{1+(2\cos^2\theta - 1)}$ **M1 M1**

$= \dfrac{2\sin\theta\cos\theta}{2\cos^2\theta}$

$= \dfrac{\sin\theta}{\cos\theta}$

$= \tan\theta$ = RHS **A1***

(b) $\tan\theta = 3$ **M1**, so θ = 71.6° **A1**

Prove an identity by combining fractions

4 marks | Grade A

1. Put the two fractions over a common denominator.
2. Expand the brackets and use $\sin^2 x + \cos^2 x = 1$
3. Factorise the numerator and cancel.
4. Write the result in the form asked for.

Example. Prove that $\dfrac{\cos x}{1+\sin x} + \dfrac{1+\sin x}{\cos x} \equiv 2\sec x$

Model answer.

$$\text{LHS} = \frac{\cos^2 x + (1+\sin x)^2}{\cos x(1+\sin x)} \qquad \textbf{M1}$$

$$= \frac{\cos^2 x + 1 + 2\sin x + \sin^2 x}{\cos x(1+\sin x)}$$

$$= \frac{2 + 2\sin x}{\cos x(1+\sin x)} \qquad \textbf{M1}$$

$$= \frac{2(1+\sin x)}{\cos x(1+\sin x)} \qquad \textbf{A1}$$

$$= \frac{2}{\cos x} = 2\sec x = \text{RHS} \qquad \textbf{A1*}$$

Prove an identity by factorising

3 marks | Grade A

1. Spot the difference of two squares.
2. Use $1 + \tan^2 x = \sec^2 x$ in both brackets.
3. Simplify to the right-hand side.

Example. Prove that $\sec^4 x - \tan^4 x \equiv 1 + 2\tan^2 x$

Model answer.

$$\text{LHS} = (\sec^2 x - \tan^2 x)(\sec^2 x + \tan^2 x) \qquad \textbf{M1}$$

$$\sec^2 x - \tan^2 x = 1 \text{ and } \sec^2 x = 1 + \tan^2 x \qquad \textbf{M1}$$

$$\text{LHS} = 1 \times (1 + \tan^2 x + \tan^2 x)$$

$$= 1 + 2\tan^2 x = \text{RHS} \qquad \textbf{A1*}$$

Prove an identity using compound angles

4 marks | Grade A*

1. Write 3x as 2x + x and use the compound angle formula.
2. Use the double angle formulae, choosing forms in sin x.
3. Replace $\cos^2 x$ with $1 - \sin^2 x$ and collect terms.

Example. Prove that $\sin 3x \equiv 3\sin x - 4\sin^3 x$

Model answer.

$$\sin 3x = \sin(2x + x)$$
$$= \sin 2x \cos x + \cos 2x \sin x \quad \text{M1}$$
$$= 2\sin x \cos^2 x + (1 - 2\sin^2 x)\sin x \quad \text{M1}$$
$$= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x \quad \text{M1}$$
$$= 3\sin x - 4\sin^3 x \quad \text{A1*}$$

Don't lose marks

- Working on both sides at once as if solving an equation, e.g. cross-multiplying.
- Skipping steps: in a 'show that' the examiner needs to see every stage.
- Expanding $(1 + \sin x)^2$ as $1 + \sin^2 x$ and missing the $2\sin x$
- Choosing a form of $\cos 2x$ that makes the algebra harder.
- Not writing a conclusion, so the proof is incomplete.

More tips

Memory tricks

- Stuck? Convert everything to sin and cos. It is not always the neatest way, but it almost always works.
- Before you start, test the identity with a value such as x = 20° on your calculator: if the two sides don't match, you have copied it wrongly.
- Look at the target: if it has only sin, replace every $\cos^2 x$ with $1 - \sin^2 x$
- $1 + \cos 2x$ and $1 - \cos 2x$ are signals to use $2\cos^2 x$ and $2\sin^2 x$

Exam technique

- Begin with 'LHS =' and end with '= RHS' (or a sentence) so that the logic is clear.
- Name the identity you use if it is not obvious.
- In a 'hence' part, you must use the result you have just proved.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade C — **Prove simple identities using $\sin^2 x + \cos^2 x = 1$**
e.g. show that $(1 - \cos x)(1 + \cos x) \equiv \sin^2 x$

Grade B — **Prove identities involving sec, cosec and cot**
Write everything in terms of sin and cos, then simplify.

Grade B — **Use double angle formulae in proofs**
Replace $\sin 2\theta$ and $\cos 2\theta$ with expressions in θ, choosing the form of $\cos 2\theta$ that simplifies best.

Grade A — **Combine fractions to prove an identity**
Use a common denominator, then simplify using $\sin^2 x + \cos^2 x = 1$ and factorising.

Grade A — **Use a proved identity to solve an equation**
Replace the complicated expression with the simpler side of the identity, then solve in the interval.

Grade A* — **Prove identities using compound angles**
e.g. prove that $\sin 3x \equiv 3\sin x - 4\sin^3 x$ by expanding $\sin(2x + x)$

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