Solving trigonometric equations

P5.7 · Pure mathematics · revision notes

You solve trigonometric equations in a given interval, in degrees or radians. These include equations with multiple angles such as $\cos 2x$, shifted angles such as $\sin(x + 30°)$, and quadratics in sin, cos or tan. Many questions need an identity first, and you must find every solution in the interval.

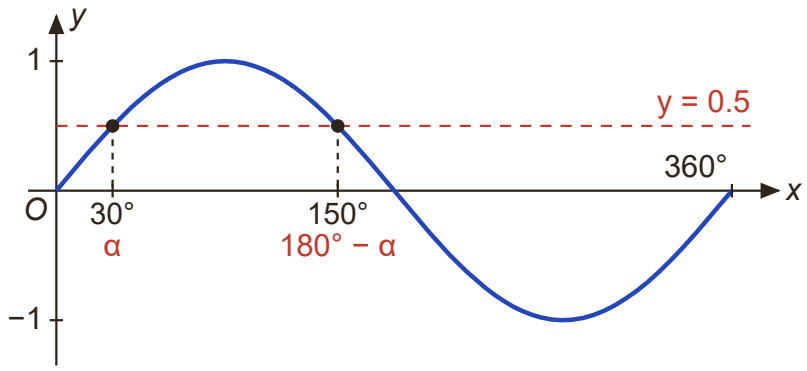
Key facts

- $\sin x = k$: $x = \alpha$ or $180° - \alpha$, then add multiples of 360°
- $\tan x = k$: $x = \alpha$, then add multiples of 180°
- For $\sin bx$: multiply the interval by b, solve, then divide by b
- $|\sin x| \leq 1$ and $|\cos x| \leq 1$; tan x can take any value
- $\cos x = k$: $x = \alpha$ or $360° - \alpha$, then add multiples of 360°
- Radians: 180° = π and 360° = 2π
- For $\sin(x + c)$: add c to both ends of the interval, solve, then subtract c

Notes

The basic method

- Find the principal value with $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ on your calculator.
- Use symmetry to find the rest: if $\sin x = k$, x = α or 180° − α; if $\cos x = k$, x = α or 360° − α; if $\tan x = k$, x = α or α + 180°. Then add or subtract 360° (180° for tan) as needed.



sin x = 0.5 has two solutions from 0° to 360°: α = 30° and 180° − α = 150°.

- In radians, 180° becomes π and 360° becomes 2π.
- The principal value may be outside the interval (e.g. negative); you still use it to find the solutions inside.
- A quick sketch of the graph with the line y = k shows how many solutions to expect.

Multiple and shifted angles

- For $\sin 3x = k$ with $0 \leq x < 360°$: let u = 3x, so $0 \leq u < 1080°$. Find every u, then divide by 3.
- For $\cos(x - 40°) = k$ with $0 \leq x < 360°$: let $u = x - 40°$, so $-40° \leq u < 320°$. Find every u, then add 40°.
- Always change the interval first, solve for u, then convert back to x.

Quadratics and identities

- Treat the trig function as the unknown: $3\tan^2 x - 5\tan x - 2 = 0$ factorises to $(3\tan x + 1)(\tan x - 2) = 0$
- If there is a mix of functions, use an identity to get just one: $\sin^2 x + \cos^2 x = 1$ or $\tan x = \dfrac{\sin x}{\cos x}$
- Other identities to use: $1 + \tan^2 x = \sec^2 x$, or a double angle formula such as $\cos 2x = 1 - 2\sin^2 x$
- Reject values outside the range of the function: $\sin x = -2$ has no solutions.

Don't lose solutions

- Don't divide both sides by a trig function that could be zero. For $\sin 2x = \sin x$, write $2\sin x \cos x - \sin x = 0$, so $\sin x(2\cos x - 1) = 0$
- Dividing $\sin x = 2\cos x$ by $\cos x$ is safe: if $\cos x = 0$ then $\sin x = \pm 1$, which does not satisfy the equation.
- If you square both sides, check each answer in the original equation, because squaring can create extra solutions.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Solve an equation with a multiple angle

4 marks **Grade C**

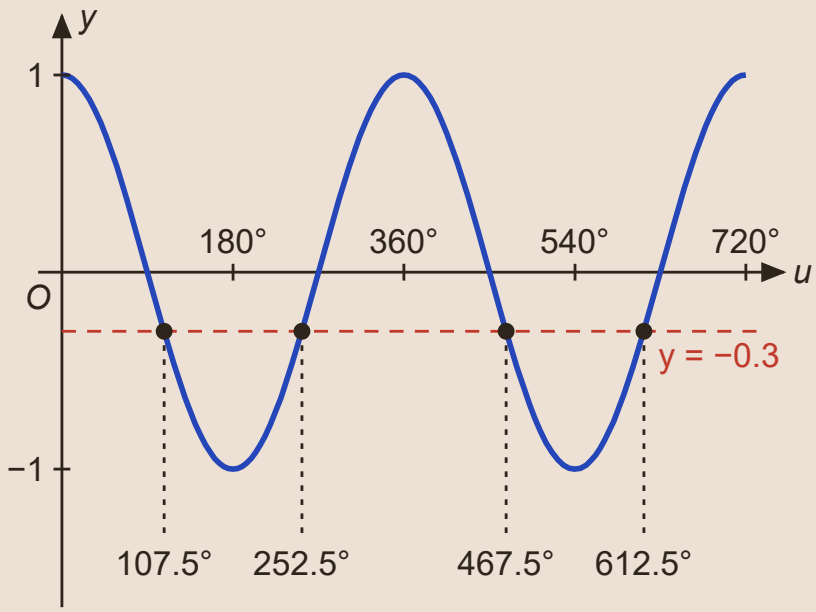
1. Replace the multiple angle with u and multiply the interval.
2. Find the principal value and use symmetry for the second value.
3. Add 360° to get every value of u in the new interval.
4. Divide by the multiple and round at the end.

Example. Solve, for $0 \leq x < 360°$, the equation $\cos 2x = -0.3$, giving your answers to 1 decimal place.

Model answer.

Let u = 2x, so $0 \leq u < 720°$ **M1**

$u = \cos^{-1}(-0.3) = 107.46°$ or $360° - 107.46° = 252.54°$ **A1**



Adding 360°: $u = 467.46°$ or $612.54°$ **M1**

x = 53.7°, 126.3°, 233.7°, 306.3° **A1**

Solve an equation with a shifted angle in radians

4 marks | Grade B

1. Make the trig function the subject.
2. Shift the interval: subtract $\frac{\pi}{4}$ from both ends.
3. Find all the values in the shifted interval; a negative value may be needed.
4. Add $\frac{\pi}{4}$ back on to each.

Example. Solve $2\cos\left(x-\frac{\pi}{4}\right)=\sqrt{3}$ for $0\le x<2\pi$, giving your answers as exact multiples of π.

Model answer.

$\cos\left(x-\frac{\pi}{4}\right)=\frac{\sqrt{3}}{2}$; let $u=x-\frac{\pi}{4}$, so $-\frac{\pi}{4}\le u<\frac{7\pi}{4}$ **M1**

$u=\frac{\pi}{6}$ **A1** or $u=-\frac{\pi}{6}$ ($\frac{11\pi}{6}$ is outside the interval) **M1**

$x=\frac{\pi}{4}-\frac{\pi}{6}=\frac{\pi}{12}$ and $x=\frac{\pi}{4}+\frac{\pi}{6}=\frac{5\pi}{12}$ **A1**

Solve a quadratic in tan x

5 marks | Grade B

1. Factorise as a quadratic in tan x.
2. Solve each factor separately.
3. For tan, add 180° to find the next solution.

Example. Solve, for $0\le x<360°$, the equation $3\tan^2 x-5\tan x-2=0$, giving your answers to 1 decimal place.

Model answer.

$(3\tan x+1)(\tan x-2)=0$ **M1**

$\tan x=2$ or $\tan x=-\frac{1}{3}$ **A1**

$\tan x=2$: x = 63.4°, 243.4° **M1**

$\tan x=-\frac{1}{3}$: the principal value is −18.4°, so x = 161.6°, 341.6° **A1 A1**

Use an identity, then solve

5 marks | Grade A

1. Replace $\cos^2 x$ with $1 - \sin^2 x$ so that only sin x is left.
2. Rearrange into a quadratic equal to zero and factorise.
3. Reject any impossible value, then solve the rest.

Example. Solve, for $0 \leq x < 360°$, the equation $2\cos^2 x = 3\sin x$

Model answer.

$2(1 - \sin^2 x) = 3\sin x$	M1
$2\sin^2 x + 3\sin x - 2 = 0$	A1
$(2\sin x - 1)(\sin x + 2) = 0$	M1
$\sin x = -2$ has no solutions, so $\sin x = \frac{1}{2}$	A1
x = 30°, 150°	A1

Solve by factorising, without dividing

4 marks | Grade A*

1. Use the double angle formula so that every term is in x.
2. Bring everything to one side and factorise; do not divide by sin x.
3. Solve each factor, including any solutions at the ends of the interval.

Example. Solve $\sin 2x = \sin x$ for $0 \leq x \leq 2\pi$. Give your answers as exact multiples of π.

Model answer.

$2\sin x \cos x = \sin x$	M1
$2\sin x \cos x - \sin x = 0 \Rightarrow \sin x(2\cos x - 1) = 0$	M1
$\sin x = 0$: x = 0, π, 2π	A1
$\cos x = \frac{1}{2}$: $x = \frac{\pi}{3}, \frac{5\pi}{3}$	A1

Don't lose marks

- Giving only the calculator value and missing the other solutions.
- Dividing by 2 (for $\sin 2x$) before finding all the solutions for 2x.
- Dividing both sides by sin x or cos x and losing solutions.
- Working in degrees when the interval is in radians, or the other way round.
- Including values outside the interval, e.g. 360° when the interval is $0 \leq x < 360°$

More tips

Memory tricks

- Sketch the graph over the (adjusted) interval and draw the line y = k: the number of crossings is the number of solutions.
- For tan, just keep adding 180° (or π).
- Check each answer by substituting it back into the original equation on your calculator.
- For $\sin kx = c$ or $\cos kx = c$ with $-1 < c < 1$, there are usually 2k solutions in a 360° interval.

Exam technique

- Unless told otherwise, give angles in degrees to 1 d.p. and in radians to 3 s.f., or as exact multiples of π when the question asks for exact answers.
- Read the interval carefully: $\leq$ or $<$, degrees or radians.
- If the question says 'Solutions relying entirely on calculator technology are not acceptable', show the algebra and each step to the angles.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Solve $\sin x = k$ in a given interval**

Find the calculator value, then use symmetry (or CAST) to find every other solution in the interval.

Grade D — **Use $\tan x = \dfrac{\sin x}{\cos x}$ to solve equations**

Turn $3\sin x = 2\cos x$ into $\tan x = \dfrac{2}{3}$ and solve.

Grade C — **Solve equations with multiple angles**

For $\cos 2x = k$, solve for 2x in the doubled interval, then halve.

Grade C — **Solve equations with a shifted angle**

For $\cos(x - \alpha) = k$, shift the interval by α, solve, then add α back on.

Grade B — **Solve quadratic equations in sin, cos or tan**

Factorise, solve each factor, and reject any impossible value.

Grade A — **Use identities before solving**

Use a Pythagorean or double angle identity to get an equation in one trig function.

Grade A* — **Factorise instead of dividing to keep solutions**

e.g. $\sin 2x = \sin x$ gives $\sin x(2\cos x - 1) = 0$, so the solutions of $\sin x = 0$ are not lost.

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