Compound and double angles; R cos(θ ± α)

P5.6 · Pure mathematics · revision notes

The compound angle formulae expand expressions such as $\sin(A+B)$, and the double angle formulae are the special case A = B. You also write $a\cos\theta + b\sin\theta$ as a single function, $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$, to solve equations and to find maximum and minimum values. These are among the most tested trigonometry skills at A level.

Key facts

- $\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$
- $\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$
- $\tan(A \pm B) \equiv \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
- $\sin 2A \equiv 2\sin A\cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2\tan A}{1-\tan^2 A}$
- $a\cos\theta + b\sin\theta \equiv R\cos(\theta - \alpha)$, where $R = \sqrt{a^2+b^2}$ and $\tan\alpha = \dfrac{b}{a}$
- Maximum and minimum of $k + R\cos(\theta - \alpha)$: $k + R$ and $k - R$

Notes

Compound angle formulae

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$

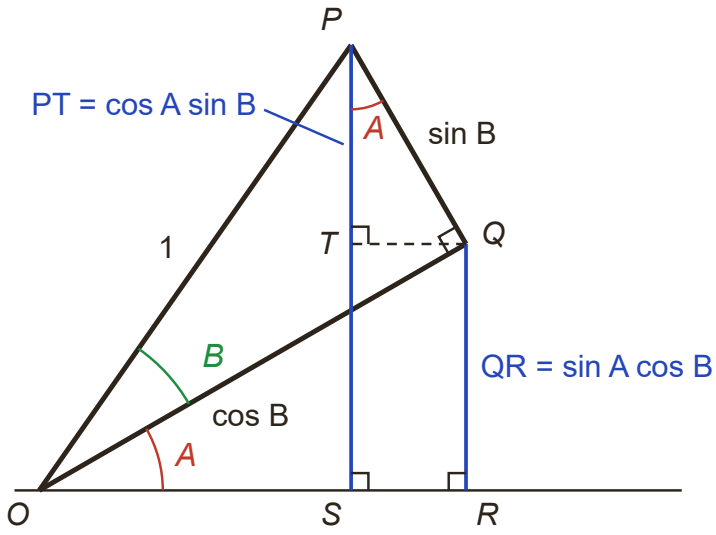
- $\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$ (the sign flips)

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

- These are in the formulae booklet. Note that $\sin(A + B)$ is **not** $\sin A + \sin B$
- Use them for exact values, e.g.

$$\begin{aligned} \cos 15^\circ &= \cos(45^\circ - 30^\circ) \\ &= \frac{\sqrt{6} + \sqrt{2}}{4} \end{aligned}$$

- You should understand how these formulae can be proved geometrically, using right-angled triangles.



The height of P is PS = QR + PT, so sin(A + B) = sin A cos B + cos A sin B.

Double angle formulae

- Put B = A in the compound angle formulae. The double angle formulae are not in the formulae booklet, so learn them.

$$\sin 2A \equiv 2 \sin A \cos A$$

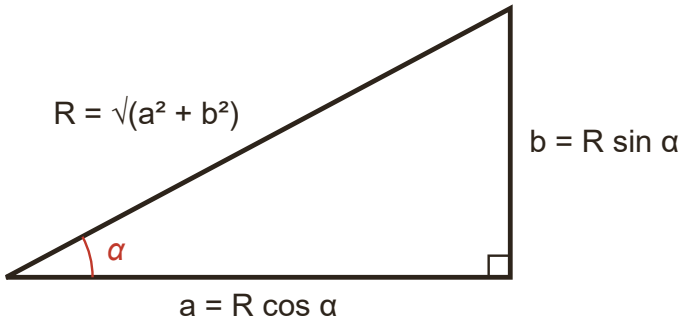
$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2 \cos^2 A - 1 \equiv 1 - 2 \sin^2 A$$

$$\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$$

- Rearranged: $\cos^2 A \equiv \frac{1}{2}(1 + \cos 2A)$ and $\sin^2 A \equiv \frac{1}{2}(1 - \cos 2A)$
- They work for any angle and half of it, e.g. $\sin x \equiv 2 \sin \frac{x}{2} \cos \frac{x}{2}$

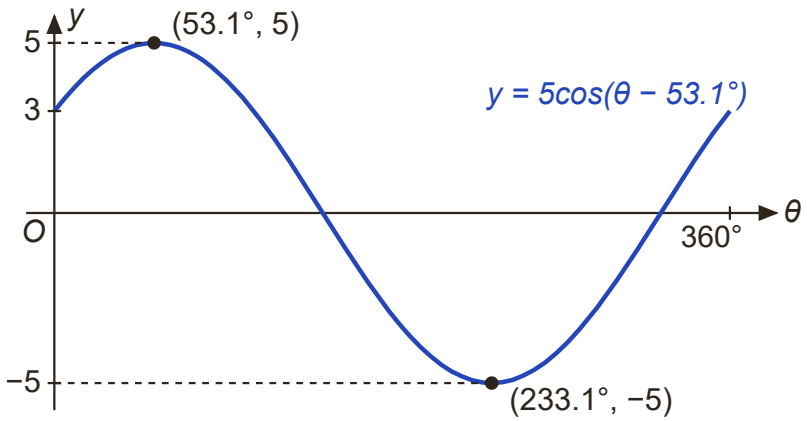
The R form

- Any expression $a\cos\theta + b\sin\theta$ can be written as $R\cos(\theta - \alpha)$, with R > 0.
- Expand: $R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$. Compare coefficients: $R\cos\alpha = a$ and $R\sin\alpha = b$
- Square and add: $R = \sqrt{a^2 + b^2}$. Divide: $\tan\alpha = \dfrac{b}{a}$



Comparing coefficients gives this triangle: R is the hypotenuse and tan α = b/a.

- The same method works for $R\cos(\theta + \alpha)$, $R\sin(\theta + \alpha)$ and $R\sin(\theta - \alpha)$: always expand the form you are given.
- The maximum of $R\cos(\theta - \alpha)$ is R, when $\theta - \alpha = 0$; the minimum is −R, when $\theta - \alpha = 180°$



3cos θ + 4sin θ = 5cos(θ − 53.1°): a cos curve with amplitude 5, moved 53.1° to the right.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, A1* = the solution must be fully correct, with every step shown.

Show an exact value using a compound angle formula

3 marks | Grade C

1. Split the angle into two angles with known exact values (30°, 45°, 60°, …).
2. Expand with the compound angle formula.
3. Substitute the exact values and simplify the surds.

Example. Without using a calculator, show that $\sin 75^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

Model answer.

$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ + 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \quad \textbf{M1}\\ &= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\times\frac{1}{2} \quad \textbf{A1}\\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4} \quad \textbf{A1*}\end{aligned}$$

Find exact values from given ratios

6 marks | Grade B

1. Draw right-angled triangles to find the other ratios of each angle.
2. Use the quadrant to fix the signs (cos is negative for an obtuse angle).
3. Substitute into the double or compound angle formula and simplify.

Example. Angle A is acute and $\tan A = \dfrac{3}{4}$. Angle B is obtuse and $\sin B = \dfrac{5}{13}$. Find the exact value of

(a) $\sin 2A$

(b) $\cos(A+B)$

Model answer.

$\sin A = \dfrac{3}{5}$ and $\cos A = \dfrac{4}{5}$ **B1**; B is obtuse, so $\cos B = -\dfrac{12}{13}$ **B1**

(a) $\sin 2A = 2\times\dfrac{3}{5}\times\dfrac{4}{5}$ **M1** $= \dfrac{24}{25}$ **A1**

(b) $\cos(A+B) = \dfrac{4}{5}\times\left(-\dfrac{12}{13}\right) - \dfrac{3}{5}\times\dfrac{5}{13}$ **M1** $= -\dfrac{63}{65}$ **A1**

Write in R form, then solve an equation 7 to 8 marks **Grade A**

1. Expand $R\cos(\theta - \alpha)$ and compare the coefficients of cos θ and sin θ.
2. $R = \sqrt{a^2 + b^2}$; find α from tan α, checking the quadrant.
3. To solve, divide by R and adjust the interval for $\theta - \alpha$
4. Find all the values in the adjusted interval, then add α back on.

Example. (a) Express $3\cos\theta + 4\sin\theta$ in the form $R\cos(\theta - \alpha)$, where R > 0 and $0 < \alpha < 90°$. Give α to 2 decimal places.
(b) Hence solve $3\cos\theta + 4\sin\theta = 2$ for $0 \leq \theta < 360°$, giving your answers to 1 decimal place.

Model answer.

(a) $R\cos\alpha = 3$ and $R\sin\alpha = 4$

$R = \sqrt{3^2 + 4^2} = 5$ **B1**

$\tan\alpha = \frac{4}{3}$ **M1**, so α = 53.13° **A1**

(b) $\cos(\theta - 53.13°) = \frac{2}{5}$ **M1**

For $-53.13° \leq \theta - 53.13° < 306.87°$: $\theta - 53.13° = 66.42°$ **A1** or $360° - 66.42° = 293.58°$ **M1**

θ = 119.6°, 346.7° **A1**

Use the R form to find a maximum or minimum 3 marks **Grade A**

1. Replace the expression with its R form.
2. Use $-R \leq R\cos(\theta - \alpha) \leq R$
3. For a fraction with a positive denominator, the largest denominator gives the smallest value.
4. Find θ by setting the angle in the bracket equal to 0° (for the maximum of cos) or 180° (for the minimum).

Example. Given that $3\cos\theta + 4\sin\theta \equiv 5\cos(\theta - 53.13°)$, find the minimum value of $\frac{20}{8 + 3\cos\theta + 4\sin\theta}$ and the smallest positive value of θ at which it occurs.

Model answer.

The fraction is smallest when the denominator is largest **M1**

Largest denominator $= 8 + 5 = 13$, so the minimum value is $\frac{20}{13}$ **A1**

This happens when $\cos(\theta - 53.13°) = 1$, so θ = 53.1° **B1**

Don’t lose marks

- Writing $\sin(A+B) = \sin A + \sin B$
- Getting the sign wrong in $\cos(A \pm B)$
- Finding α from $\tan\alpha = \dfrac{a}{b}$ instead of $\dfrac{b}{a}$; always compare coefficients.
- Writing R = 25 instead of $\sqrt{25} = 5$, or using $\sqrt{a^2 - b^2}$
- Not adjusting the interval for $\theta - \alpha$, so solutions are lost or extra ones are included.
- Rounding α too early, so the final answers come out wrong.

More tips

Memory tricks

- 'Cos is contrary': in $\cos(A+B)$ the sign in the middle becomes minus, and in $\cos(A-B)$ it becomes plus.
- For $R\cos(\theta - \alpha)$, $\tan\alpha$ is the coefficient of sin θ divided by the coefficient of cos θ. For the other forms, expand and compare rather than memorising.
- Check your R form by putting θ = 0: $R\cos(-\alpha) = R\cos\alpha$ should equal a.
- Choose the form of $\cos 2A$ that matches the rest of the equation: $1 - 2\sin^2 A$ if it has sin A, $2\cos^2 A - 1$ if it has cos A.

Exam technique

- Give R exactly (e.g. $\sqrt{13}$) and α to the accuracy asked, often 2 d.p. in degrees or 3 d.p. in radians.
- 'Hence' means use the R form you have just found.
- When a question asks for the maximum and where it occurs, give both the value and θ.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade C — **Find exact values using compound angles**

e.g.

$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ + 30^\circ) \\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Grade C — **Use the double angle formulae**

Know $\sin 2A$, the three forms of $\cos 2A$ and $\tan 2A$, and choose the most useful form.

Grade B — **Use given ratios to find compound values**

e.g. given tan A and sin B, find $\cos(A+B)$ or $\sin 2A$ exactly, taking care with signs.

Grade B — **Write $a\cos\theta + b\sin\theta$ in R form**

Expand $R\cos(\theta - \alpha)$ and compare coefficients: $R = \sqrt{a^2 + b^2}$ and $\tan\alpha = \dfrac{b}{a}$

Grade A

Solve equations using the R form

Rewrite as $R\cos(\theta - \alpha) = c$, adjust the interval for $\theta - \alpha$, solve and convert back.

Grade A

Find maximum and minimum values using R form

The maximum of $R\cos(\theta - \alpha)$ is R and the minimum is −R; use this in fractions and in models.

Grade A*

Derive new identities from compound angles

e.g. write $\sin 3x$ in terms of $\sin x$ by expanding $\sin(2x + x)$

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