# Pythagorean identities

P5.5 · Pure mathematics · revision notes

The identities $\sin^2\theta + \cos^2\theta \equiv 1$, $1 + \tan^2\theta \equiv \sec^2\theta$ and $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$, together with $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$, let you find exact values, solve equations and prove other identities. They appear in almost every trigonometry question on Papers 1 and 2.

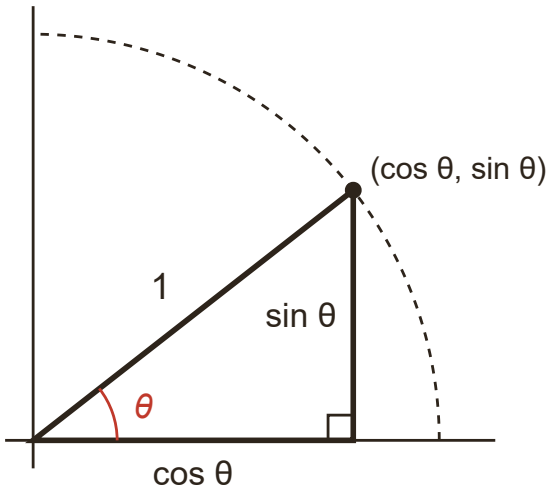
## Key facts

- $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$
- $\sin^2\theta + \cos^2\theta \equiv 1$
- $1 + \tan^2\theta \equiv \sec^2\theta$
- $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$
- $\cos^2\theta \equiv 1 - \sin^2\theta$ and $\sin^2\theta \equiv 1 - \cos^2\theta$
- $-1 \le \sin\theta \le 1$, $-1 \le \cos\theta \le 1$, $|\sec\theta| \ge 1$, $|\operatorname{cosec}\theta| \ge 1$

## Notes

### The identities

- $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ (where $\cos\theta \neq 0$).
- $\sin^2\theta + \cos^2\theta \equiv 1$. This is Pythagoras' theorem for the point $(\cos\theta, \sin\theta)$ on the unit circle.

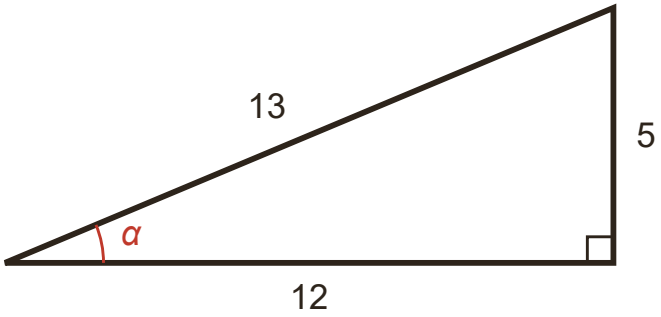



Pythagoras in this triangle gives cos²θ + sin²θ = 1.

- Divide every term by $\cos^2\theta$: $\tan^2\theta + 1 \equiv \sec^2\theta$
- Divide every term by $\sin^2\theta$: $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$
- Useful rearrangements: $\sin^2\theta \equiv 1 - \cos^2\theta$ and $\cos^2\theta \equiv 1 - \sin^2\theta$
- Also $\tan^2\theta \equiv \sec^2\theta - 1$ and $\cot^2\theta \equiv \operatorname{cosec}^2\theta - 1$
- $\sin^2\theta$ means $(\sin\theta)^2$. The symbol ≡ means the statement is true for every value of θ for which it is defined.

## Exact values

- Given one ratio, use an identity to find another, then choose the sign from the quadrant (CAST).
- Example: if $\sin\theta = -\frac{5}{13}$ and $180° < \theta < 270°$, then $\cos^2\theta = 1 - \frac{25}{169} = \frac{144}{169}$. cos is negative in this quadrant, so $\cos\theta = -\frac{12}{13}$
- You can also draw a right-angled triangle for the acute angle (here with sides 5, 12 and 13), then add the signs.




For the acute angle α: sin α = 5/13, cos α = 12/13, tan α = 5/12. Then add the signs for the quadrant.

## Solving equations

- Aim for an equation in **one** trig function: replace $\cos^2 x$ with $1 - \sin^2 x$ if the equation also has sin x.
- Replace $\tan^2 x$ with $\sec^2 x - 1$ if the equation also has sec x.
- Rearrange into a quadratic equal to 0, factorise (or use the formula), then solve each factor in the given interval.
- Reject impossible values: $\sin x = 2$ and $\cos x = -3$ have no solutions.
- $\sec x = \frac{1}{2}$ has no solutions either, because $|\sec x| \geq 1$
- Change sec, cosec or cot into cos, sin or tan before using your calculator.

## How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, A1* = the solution must be fully correct, with every step shown, ft = follow through: you still earn the mark if you use your own earlier wrong answer correctly.

## Find exact values from one given ratio

3 marks | Grade C

1. Use $\sin^2\theta + \cos^2\theta = 1$ to find the square of the missing ratio.
2. Choose the sign using the quadrant.
3. Use $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ to find tan θ.

**Example.** Given that $\sin\theta = -\dfrac{5}{13}$ and $180° < \theta < 270°$, find the exact value of

(a) $\cos\theta$

(b) $\tan\theta$

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**Model answer.**

(a)
$$\cos^2\theta = 1 - \left(-\frac{5}{13}\right)^2$$
$$= \frac{144}{169} \qquad \textbf{M1}$$

θ is in the third quadrant, so $\cos\theta = -\dfrac{12}{13}$ **A1**

(b) $\tan\theta$
$$= \frac{-\frac{5}{13}}{-\frac{12}{13}}$$
$$= \frac{5}{12} \qquad \textbf{B1ft}$$

## Solve a quadratic equation using $\sin^2 x + \cos^2 x = 1$

5 marks | Grade B

1. Replace $\cos^2 x$ with $1 - \sin^2 x$ so that only sin x is left.
2. Rearrange into a quadratic equal to zero and factorise.
3. Solve each factor in the interval, rejecting any impossible value.

**Example.** Solve, for $0 \le x < 360°$, the equation $2\cos^2 x = 1 + \sin x$

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**Model answer.**

$2(1 - \sin^2 x) = 1 + \sin x$ **M1**

$2\sin^2 x + \sin x - 1 = 0$ **A1**

$(2\sin x - 1)(\sin x + 1) = 0$, so $\sin x = \dfrac{1}{2}$ or $\sin x = -1$ **M1 A1**

x = 30°, 150°, 270° **A1**

**Solve an equation using $1 + \tan^2\theta = \sec^2\theta$**

5 marks | Grade A

1. Replace $\tan^2\theta$ with $\sec^2\theta - 1$
2. Solve the quadratic in sec θ.
3. Turn each value of sec θ into a value of cos θ.
4. Find every angle in the interval, including 0° if it is allowed.

**Example.** Solve, for $0 \leq \theta < 360°$, the equation $2\tan^2\theta + \sec\theta = 1$. Give your answers to one decimal place where appropriate.

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**Model answer.**

$2(\sec^2\theta - 1) + \sec\theta = 1$ **M1**

$2\sec^2\theta + \sec\theta - 3 = 0$ **A1**

$(2\sec\theta + 3)(\sec\theta - 1) = 0$, so $\cos\theta = 1$ or $\cos\theta = -\frac{2}{3}$ **M1 A1**

θ = 0°, 131.8°, 228.2° **A1**

**Eliminate θ to find an equation in x and y**

3 marks | Grade A*

1. Make each trig function the subject.
2. Choose the identity that links the two functions.
3. Substitute and rearrange into the required form.

**Example.** Given that $x = 2\,\text{cosec}\,\theta$ and $y = 5\cot\theta$, show that $25x^2 - 4y^2 = 100$

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**Model answer.**

$\text{cosec}\,\theta = \frac{x}{2}$ and $\cot\theta = \frac{y}{5}$

Using $1 + \cot^2\theta = \text{cosec}^2\theta$: $1 + \frac{y^2}{25} = \frac{x^2}{4}$ **M1 M1**

Multiply by 100: $100 + 4y^2 = 25x^2$, so $25x^2 - 4y^2 = 100$ **A1***

## Don't lose marks

- Writing $\cos^2 x = 1 - \sin x$ (the square is lost).
- Sign slips: $\sec^2\theta = 1 + \tan^2\theta$, not $1 - \tan^2\theta$
- Giving a positive value for cos θ when θ is in the second or third quadrant.
- Not rejecting impossible values such as $\sin x = -2$
- Missing solutions at the ends of the interval, such as θ = 0°.

## More tips

### Memory tricks

- Learn $\sin^2\theta + \cos^2\theta = 1$ well: divide it by $\cos^2\theta$ to get the tan and sec identity, or by $\sin^2\theta$ to get the cot and cosec one.
- Partners: tan goes with sec, and cot goes with cosec.
- Quick check with θ = 45°: $\sec^2 45° = 2 = 1 + \tan^2 45°$
- The 3, 4, 5 and 5, 12, 13 triangles give exact values quickly.

### Exam technique

- Show the line where you substitute the identity; it usually earns the first method mark.
- Give angles to 1 d.p. in degrees (or 3 s.f. in radians) unless you are told otherwise.
- When the question says 'exact', leave answers as fractions or surds.

## What each grade needs

What you need to be able to do, from the first marks up to the top grade.

**Grade D** — **Use $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ to simplify**
e.g. $\sin\theta = 2\cos\theta$ becomes $\tan\theta = 2$

**Grade C** — **Find exact values using $\sin^2\theta + \cos^2\theta = 1$**
Given one ratio and the quadrant, find the other ratios with the correct signs.

**Grade B** — **Derive the sec and cosec identities**
Divide $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$, or by $\sin^2\theta$

**Grade B** — **Solve quadratic equations using an identity**
Replace $\cos^2 x$ by $1 - \sin^2 x$ (or similar) to get a quadratic in one function.

**Grade A** — **Solve equations with sec and tan together**
Use $\tan^2\theta = \sec^2\theta - 1$ to get a quadratic in sec θ, then change to cos θ.

**Grade A*** — **Eliminate θ between two equations**
e.g. from $x = 2\operatorname{cosec}\theta$ and $y = 5\cot\theta$, use $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

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