Small angle approximations

P5.2 · Pure mathematics · revision notes

For small angles measured in radians, sin, cos and tan can be replaced by simple expressions in θ. You use these approximations to simplify expressions, to estimate values and in geometry problems. Questions are usually short 'show that' or 'find an approximation' parts.

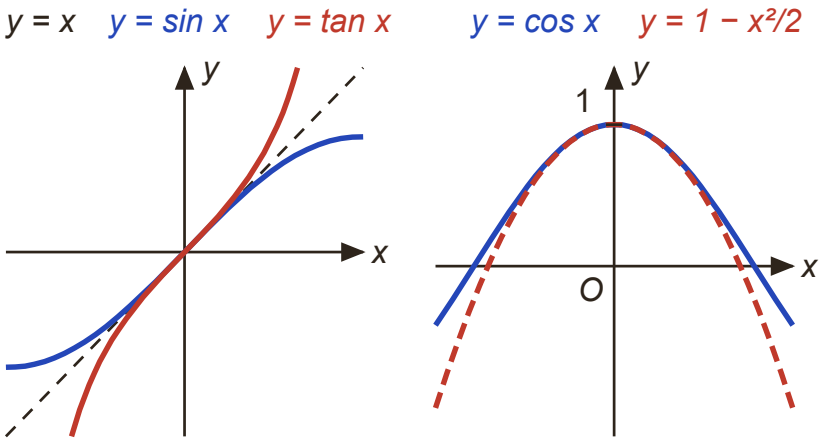
Key facts

- $\sin\theta \approx \theta$
- $\cos\theta \approx 1 - \frac{\theta^2}{2}$
- $\cos k\theta \approx 1 - \frac{k^2\theta^2}{2}$
- $\tan\theta \approx \theta$
- Only for small θ measured in radians
- $1 - \cos\theta \approx \frac{\theta^2}{2}$

Notes

The approximations

- When θ is small and measured in **radians**: $\sin\theta \approx \theta$, $\tan\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{\theta^2}{2}$
- They are given in the formulae booklet.
- Why they work: near the origin the graphs of $y = \sin x$ and $y = \tan x$ are very close to the line $y = x$, and the graph of $y = \cos x$ is very close to the curve $y = 1 - \frac{x^2}{2}$



Near x = 0 (in radians) the curves are almost on top of each other, which is why the approximations work.

- They only work in radians. For example, 5° is 0.0873 radians and $\sin 5° = 0.0872$, but $\sin 5°$ is nowhere near 5.
- The bigger θ is, the less accurate the approximations become.

Using them

- Replace every trig function, then simplify algebraically.
- With a multiple angle, replace θ by the whole angle: $\sin 4\theta \approx 4\theta$ and $\tan \frac{\theta}{2} \approx \frac{\theta}{2}$
- For cos, square the whole angle: $\cos 2\theta \approx 1 - \frac{(2\theta)^2}{2} = 1 - 2\theta^2$
- A useful rearrangement: $1 - \cos\theta \approx \frac{\theta^2}{2}$, so $1 - \cos 3\theta \approx \frac{9\theta^2}{2}$
- For a fraction, approximate the top and the bottom separately and cancel common powers of θ.
- Don't simplify $\cos\theta$ to just 1 when the θ^2 term is needed for the answer.

Accuracy

- Percentage error $= \frac{|\text{approximate} - \text{exact}|}{\text{exact}} \times 100$
- For example, $\cos 0.2 \approx 1 - \frac{0.2^2}{2} = 0.98$, while the exact value is 0.98007 (to 5 d.p.): a very small error.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, A1* = the solution must be fully correct, with every step shown.

Show that an expression is approximately equal to a given form 3 marks **Grade B**

1. Write down the approximation for each trig term, using the whole angle.
2. Substitute and simplify the numerator.
3. Divide through by the denominator to reach the given answer.

Example. Given that θ is small and measured in radians, show that $\frac{\sin 4\theta + \cos 2\theta - 1}{2\theta} \approx 2 - \theta$

Model answer.

$\sin 4\theta \approx 4\theta$ and $\cos 2\theta \approx 1 - \frac{(2\theta)^2}{2} = 1 - 2\theta^2$ **M1**

$\frac{4\theta + 1 - 2\theta^2 - 1}{2\theta} = \frac{4\theta - 2\theta^2}{2\theta}$ **M1**

$= 2 - \theta$ **A1***

Find the approximate value of a fraction

3 marks **Grade A**

1. Approximate the numerator and the denominator separately.
2. Both should give the same power of θ.
3. Cancel to get a number.

Example. Given that θ is small and measured in radians, find the approximate value of $\dfrac{1-\cos 3\theta}{\theta \sin 2\theta}$

Model answer.

$1-\cos 3\theta \approx 1-\left(1-\dfrac{9\theta^2}{2}\right)=\dfrac{9\theta^2}{2}$ **M1**

$\theta \sin 2\theta \approx \theta \times 2\theta = 2\theta^2$ **M1**

$\dfrac{9\theta^2}{2} \div 2\theta^2 = \dfrac{9}{4}$ **A1**

Estimate a value and find the percentage error

3 marks **Grade B**

1. Substitute the value into the approximation.
2. Find the exact value with your calculator in radian mode.
3. Percentage error = difference ÷ exact value × 100.

Example. (a) Use a small angle approximation to estimate the value of $\cos 0.2$.
(b) Find the percentage error in this estimate.

Model answer.

(a) $\cos 0.2 \approx 1-\dfrac{0.2^2}{2}=0.98$ **B1**

(b) $\cos 0.2 = 0.980067\ldots$ (radians) **M1**

Percentage error $= \dfrac{0.980067-0.98}{0.980067} \times 100$

$= 0.0068$ % (2 s.f.) **A1**

Use small angles in a triangle

3 marks | Grade A*

1. Use the cosine rule (or another triangle formula) to get an expression containing a trig function of θ.
2. Replace the trig function with its small angle approximation.
3. Expand and simplify to the given form.

Example. In triangle ABC, AB = 4 cm, AC = 5 cm and angle BAC = θ radians. Given that θ is small, show that $BC^2 \approx 1 + 20\theta^2$

Model answer.

$$BC^2 = 4^2 + 5^2 - 2 \times 4 \times 5\cos\theta$$
$$= 41 - 40\cos\theta \quad \text{M1}$$

$$\approx 41 - 40\left(1 - \frac{\theta^2}{2}\right) \quad \text{M1}$$

$$= 41 - 40 + 20\theta^2 = 1 + 20\theta^2 \quad \text{A1*}$$

Don't lose marks

- Using the approximations with θ in degrees.
- Writing $\cos 2\theta \approx 1 - \frac{2\theta^2}{2}$ instead of $1 - \frac{(2\theta)^2}{2} = 1 - 2\theta^2$
- Replacing $\cos\theta$ by 1 when the answer depends on the θ^2 term.
- Sign errors when subtracting: $1 - \left(1 - \frac{9\theta^2}{2}\right) = +\frac{9\theta^2}{2}$

More tips

Memory tricks

- For cos: 'one minus half the square' of the angle.
- sin and tan share the same approximation, θ, because both graphs hug y = x near the origin.
- Check an answer by putting θ = 0.01 into both sides on your calculator (in radian mode): they should be very close.
- If the question has $\sin 3\theta$, write 3θ straight away; if it has $\cos 3\theta$, write $1 - \frac{9\theta^2}{2}$

Exam technique

- Write down each approximation you use before substituting; each one is often a method mark.
- In 'show that' questions the answer is given, so show enough steps for every line to follow from the one before.
- Give exact numbers such as $\frac{9}{4}$, not decimals.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade C

Recall the three small angle approximations

$\sin\theta \approx \theta$, $\tan\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{\theta^2}{2}$, for small θ in radians.

Grade B

Approximate an expression for small θ

Replace each trig function with its approximation and simplify to an expression in θ.

Grade B

Replace θ by a multiple such as 3θ

$\sin 3\theta \approx 3\theta$ and $\cos 3\theta \approx 1 - \frac{(3\theta)^2}{2} = 1 - \frac{9\theta^2}{2}$

Grade B

Estimate a value and its percentage error

Substitute a small angle into the approximation and compare with the calculator value in radian mode.

Grade A

Find the approximate value of a fraction

Approximate the top and bottom separately, then cancel the powers of θ, e.g. to get $\frac{9}{4}$

Grade A*

Use approximations in a geometry problem

Combine them with the cosine rule or with area formulae when an angle in a diagram is small.

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