Sine and cosine rules, radians, arcs and sectors

P5.1 · Pure mathematics · revision notes

This subtopic covers the sine rule, cosine rule and area formula for any triangle, and measuring angles in radians, which gives simple formulae for arc length and sector area. Sine, cosine and tangent are also defined for angles of any size using the unit circle. These skills are used throughout Papers 1 and 2 and in mechanics.

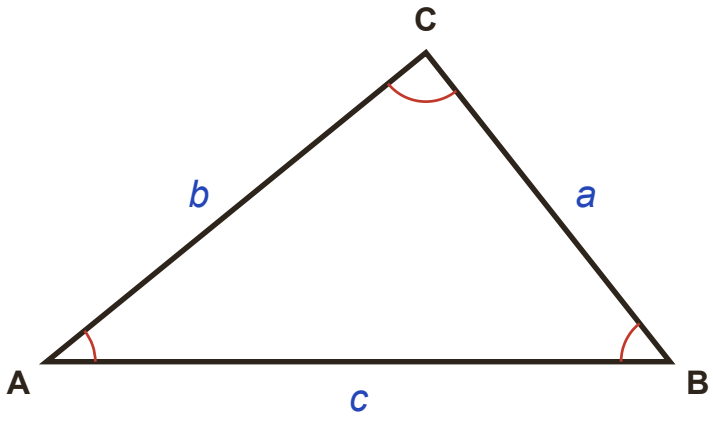
Key facts

- $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- $a^2 = b^2 + c^2 - 2bc\cos A$
- Area of a triangle $= \frac{1}{2}ab\sin C$
- π radians = 180°
- Arc length $s = r\theta$ (θ in radians)
- Sector area $= \frac{1}{2}r^2\theta$ (θ in radians)
- Segment area $= \frac{1}{2}r^2(\theta - \sin\theta)$
- Unit circle: the point at angle θ is $(\cos\theta, \sin\theta)$

Notes

Sine rule, cosine rule and area

- Label the triangle so that side a is opposite angle A, side b is opposite angle B, and so on.

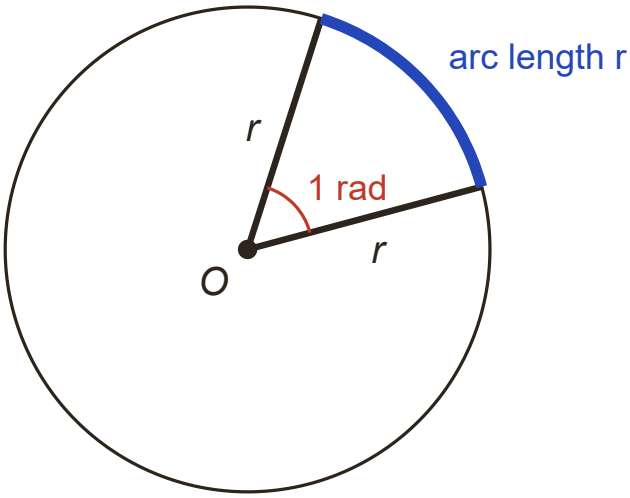


Each side takes the small letter of the angle opposite it.

- Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. Use it when you know a side and the angle opposite it.
- Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$, or $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$. Use it for two sides and the included angle, or for three sides.
- Area $= \frac{1}{2}ab\sin C$, where C is the angle between sides a and b.
- Ambiguous case: when you use the sine rule to find an angle, $\sin B = k$ has two solutions between 0° and 180°, B and 180° − B. The obtuse one is only possible if it, added to the angle you already know, is less than 180°.

Radians

- One radian is the angle at the centre of a circle made by an arc whose length equals the radius.



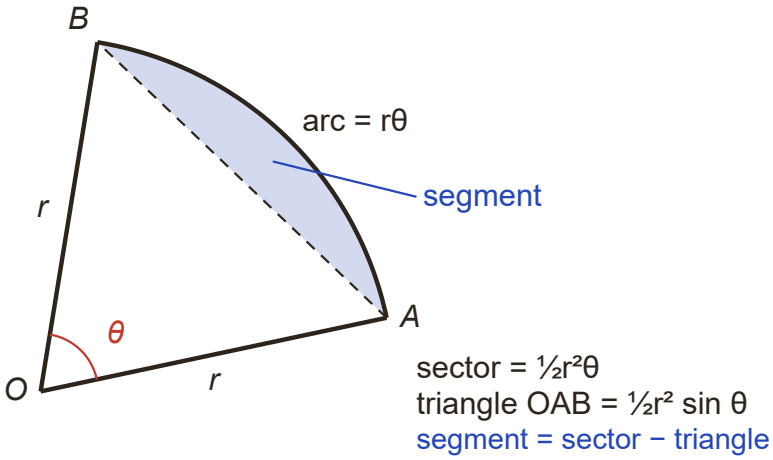
When the arc length equals the radius, the angle at the centre is 1 radian (about 57.3°).

- π radians = 180°, so 2π radians = 360° and 1 radian ≈ 57.3°.
- Degrees to radians: multiply by $\frac{\pi}{180}$. Radians to degrees: multiply by $\frac{180}{\pi}$
- Learn: $30° = \frac{\pi}{6}$, $45° = \frac{\pi}{4}$, $60° = \frac{\pi}{3}$, $90° = \frac{\pi}{2}$
- Set your calculator to radian mode whenever the angles are in radians.

Arcs, sectors and segments (θ in radians)

- Arc length $s = r\theta$
- Sector area $A = \frac{1}{2}r^2\theta$
- Perimeter of a sector $= 2r + r\theta$
- Segment area = sector − triangle

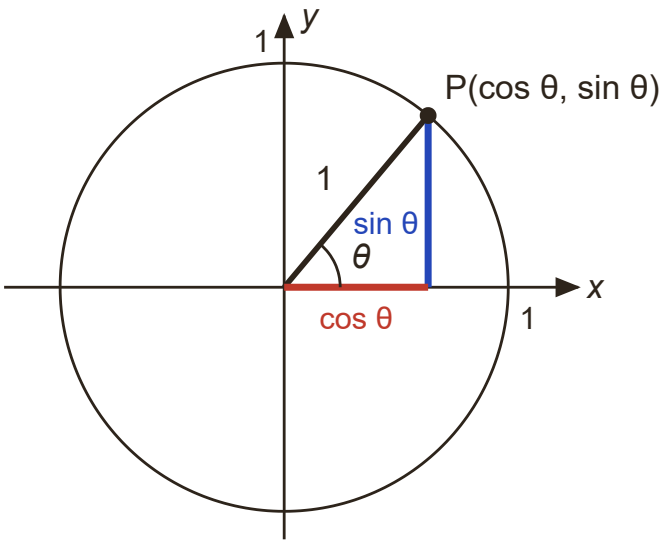
$$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin\theta$$
$$= \frac{1}{2}r^2(\theta - \sin\theta)$$



The segment is the sector minus the triangle OAB.

Trig ratios for any angle

- Measure θ anticlockwise from the positive x-axis. The point on the unit circle at angle θ is $(\cos\theta, \sin\theta)$, and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$



On the unit circle, the x-coordinate of P is cos θ and the y-coordinate is sin θ, for any angle θ.

- This defines sin, cos and tan for any angle, including obtuse, reflex and negative angles, and gives their signs in each quadrant.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, A1* = the solution must be fully correct, with every step shown.

Find a side and the area of a triangle 4 marks **Grade D**

1. Label the sides and angles, and decide which rule fits.
2. Substitute carefully. The cosine of an obtuse angle is negative, so the $-2bc\cos A$ term adds on.
3. Take the square root at the end and round to 3 significant figures.

Example. In triangle PQR, PQ = 6 cm, PR = 9 cm and angle QPR = 110°.
(a) Find the length QR.
(b) Find the area of the triangle.

Model answer.

(a) $QR^2 = 6^2 + 9^2 - 2 \times 6 \times 9\cos 110°$ **M1**

$= 153.94\ldots$, so QR = 12.4 cm (3 s.f.) **A1**

(b) Area $= \dfrac{1}{2} \times 6 \times 9 \times \sin 110°$ **M1** = 25.4 cm^2 (3 s.f.) **A1**

Find two possible angles (the ambiguous case)

3 to 4 marks **Grade C**

1. Use the sine rule with the angle you want on top.
2. Find the calculator (acute) value.
3. The other possibility is 180° minus it.
4. Check that each angle leaves room for the third angle.

Example. In triangle ABC, AB = 10 cm, BC = 7 cm and angle BAC = 40°. Find the two possible sizes of angle ACB, to one decimal place.

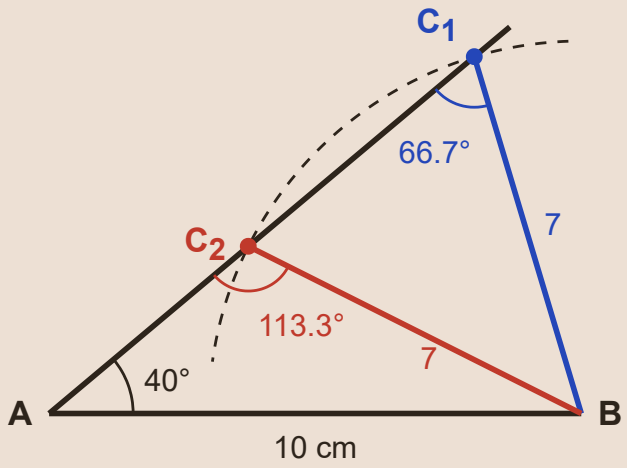
Model answer.

$$\frac{\sin C}{10} = \frac{\sin 40°}{7}$$ **M1**

$\sin C = 0.9182\ldots$, so C = 66.7° **A1**

or C = 180° − 66.7° **M1** = 113.3° **A1**

(Both are possible, because 40° + 113.3° is less than 180°.)



Find the arc length, perimeter and area of a sector

4 marks | Grade D

1. Check that the angle is in radians (convert it if it is in degrees).
2. Arc length is $r\theta$; the perimeter adds the two radii.
3. Area is $\frac{1}{2}r^2\theta$

Example. The sector OAB of a circle has centre O, radius 8 cm and angle AOB = 1.2 radians.
(a) Find the perimeter of the sector.
(b) Find the area of the sector.

Model answer.

(a) Arc AB $= 8 \times 1.2 = 9.6$ cm **M1**

Perimeter $= 8 + 8 + 9.6 = 25.6$ cm **A1**

(b) Area $= \frac{1}{2} \times 8^2 \times 1.2$ **M1** = 38.4 cm^2 **A1**

Find the area of a segment

3 marks | Grade B

1. Find the area of the sector.
2. Find the area of the triangle with $\frac{1}{2}r^2 \sin\theta$ (calculator in radian mode).
3. Subtract.

Example. A sector OAB has radius 8 cm and angle AOB = 1.2 radians, so its area is 38.4 cm^2. Find the area of the segment between the chord AB and the arc AB.

Model answer.

Triangle OAB $= \frac{1}{2} \times 8^2 \times \sin 1.2$

$= 29.825\ldots$ **M1**

Segment $= 38.4 - 29.825\ldots$ **M1** = 8.57 cm^2 (3 s.f.) **A1**

Show that an angle satisfies an equation 3 marks **Grade A**

1. Write each area or length as an expression in r and θ.
2. Form the equation from the information given.
3. Simplify step by step to the given result; r cancels.

Example. A sector of a circle has radius r and angle θ radians. The area of the segment cut off by the chord is one quarter of the area of the sector. Show that $3\theta = 4\sin\theta$

Model answer.

Segment $= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$ **M1**

$\frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{4} \times \frac{1}{2}r^2\theta$ **M1**

Divide by $\frac{1}{2}r^2$: $\theta - \sin\theta = \frac{1}{4}\theta$, so $\sin\theta = \frac{3}{4}\theta$, giving $3\theta = 4\sin\theta$ **A1***

Don’t lose marks

- Leaving the calculator in degree mode for a radian question (or the other way round).
- Using $s = r\theta$ or $\frac{1}{2}r^2\theta$ with θ in degrees.
- Missing the obtuse angle in the ambiguous case, or giving it when it does not fit in the triangle.
- Working out $(b^2 + c^2 - 2bc)\cos A$ instead of $b^2 + c^2 - 2bc\cos A$
- Trying to use the sine rule without a known side and opposite angle pair.
- Rounding lengths early and then using them again.

More tips

Memory tricks

- 180° goes with π: to convert, multiply by $\frac{\pi}{180}$ or by $\frac{180}{\pi}$
- A sector of angle θ is $\frac{\theta}{2\pi}$ of the whole circle, which is where $r\theta$ and $\frac{1}{2}r^2\theta$ come from.
- With three sides, use the cosine rule to find the largest angle first (it may be obtuse); the other two angles must then be acute, so the sine rule is safe.
- The longest side is opposite the largest angle: a quick check on any answer.

Exam technique

- Give answers to 3 significant figures unless told otherwise, with units (cm, cm^2).
- If the question asks for an exact answer, leave π and surds in your answer.
- For 'show that' results, write every step; the final line must match the given answer exactly.
- Draw or label a diagram before choosing a rule.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Use the sine and cosine rules**
Choose the rule from what you know: a side and its opposite angle means the sine rule; two sides and the included angle, or three sides, means the cosine rule.

Grade E — **Convert between degrees and radians**
Use π radians = 180°, e.g. $60° = \frac{\pi}{3}$ and $\frac{3\pi}{4} = 135°$

Grade D — **Find the area of a triangle**
Use $\frac{1}{2}ab\sin C$, where C is the angle between the sides a and b.

Grade D — **Find arc length and sector area**
Use $s = r\theta$ and $A = \frac{1}{2}r^2\theta$, with θ in radians.

Grade C — **Find both angles in the ambiguous case**
If the sine rule gives $\sin B = k$, the angle could be B or 180° − B; check whether both fit in the triangle.

Grade B — **Find the area of a segment**
Segment = sector − triangle $= \frac{1}{2}r^2(\theta - \sin\theta)$

Grade A — **Form equations from sector and triangle facts**
Combine the formulae to show that θ satisfies an equation such as $3\theta = 4\sin\theta$

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