Modelling with sequences and series

P4.6 · Pure mathematics · revision notes

Arithmetic and geometric sequences model real situations: pay that rises by a fixed amount, savings with compound interest, populations, depreciation and bouncing balls. You choose the right model, decide whether you need a term or a sum, and comment on the limitations of the model.

Key facts

- Same amount each time → arithmetic: $u_n = a + (n-1)d$
- p% increase: multiply by $1 + \frac{p}{100}$; p% decrease: multiply by $1 - \frac{p}{100}$
- Total over n periods: S_n
- Same percentage each time → geometric: $u_n = ar^{n-1}$
- Value after n years from a starting value V: $V \times r^n$
- Limit of a total that keeps shrinking: $S_\infty = \frac{a}{1-r}$

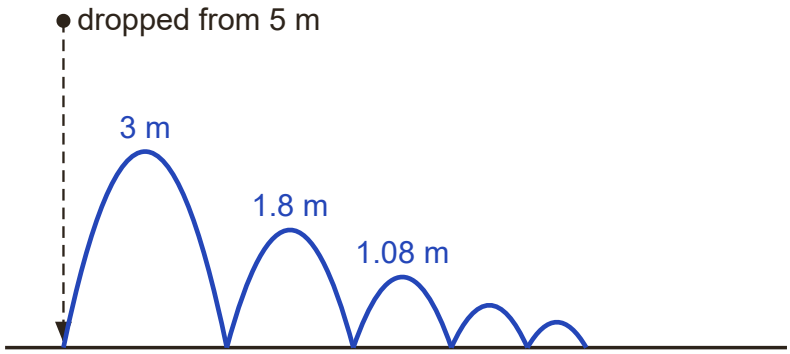
Notes

Choosing the model

- Increases or decreases by the same **amount** each time: arithmetic, with common difference d.
- Increases or decreases by the same **percentage** each time: geometric. The multiplier is $1 + \frac{p}{100}$ for a p% increase and $1 - \frac{p}{100}$ for a p% decrease.
- A 15% decrease means multiplying by 0.85, not by 0.15.
- Decide what n = 1 means. If a car is worth £18000 when new, its value after n years is 18000×0.85^n, which is the (n + 1)th term if the first term is 18000.
- A quick table of the first two or three values stops off-by-one errors.

Term or sum?

- The amount in one particular year, week or bounce is a **term**.
- The total over several years or weeks is a **sum**, S_n
- When the amounts keep shrinking by the same factor ($|r| < 1$), S_∞ gives the limit of the total, e.g. the total distance travelled by a bouncing ball.

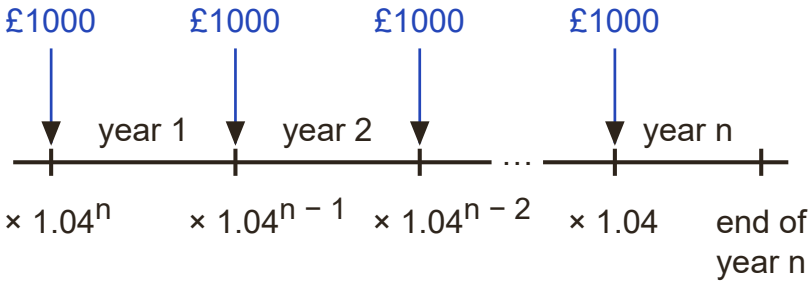


Each bounce height is 60% of the one before: 3, 1.8, 1.08, … Each of these is travelled twice, up and down.

- To find when a total or value passes a target, solve an inequality in n and give a whole-number answer in context.

Savings with compound interest

- If money is paid in at the start of every year and interest is added at the end of each year, the first payment earns interest for n years, the second for n − 1 years, and so on.
- For £1000 a year at 4%, the total at the end of year n is $1040 + 1040(1.04) + \cdots + 1040(1.04)^{n-1}$: a geometric series with first term 1040 and common ratio 1.04.



Each payment is multiplied by 1.04 once for every year it is in the account.

- Work out years 1 and 2 by hand first to check that you have the right first term.

Limitations

- Real rates change: interest rates, pay rises and depreciation are rarely exactly constant.
- A geometric model with r greater than 1 grows without limit, but real populations are limited by space and food.
- Some quantities must be whole numbers (people, animals), but the model gives decimals.
- A bouncing-ball model predicts infinitely many bounces, but a real ball stops after a finite number.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Arithmetic model: a term and a total

4 marks | Grade C

1. Identify a and d from the context.
2. Use u_n for a single week or year and S_n for a total.
3. Give units.

Example. Priya follows a training plan. She runs 3 km in week 1 and increases the distance by 0.5 km each week.
(a) Find the distance she runs in week 12.
(b) Find the total distance she runs in the first 12 weeks.

Model answer.

(a) $3 + 11 \times 0.5$ **M1** $= 8.5$ km **A1**

(b) $S_{12} = \frac{12}{2}(3 + 8.5)$ **M1** $= 69$ km **A1**

Geometric model: find a value and when a target is reached

5 marks | Grade B

1. Write down the multiplier (0.85 for a 15% fall).
2. Value after n years = starting value × multiplier to the power n.
3. Set up the inequality and take logs, reversing the sign when you divide by the log of a number below 1.
4. Give a whole-number answer in context.

Example. A car is bought for £18000. Its value falls by 15% each year.
(a) Show that its value after 3 years is £11054 to the nearest pound.
(b) Find the number of complete years after which the value of the car first falls below £5000.

Model answer.

(a) $18000 \times 0.85^3 = 11054.25$ **M1**, which is £11054 to the nearest pound **A1**

(b) $18000 \times 0.85^n < 5000 \Rightarrow 0.85^n < \frac{5}{18}$ **M1**

$n > \frac{\log\left(\frac{5}{18}\right)}{\log 0.85} = 7.88\ldots$ (the sign reverses because $\log 0.85 < 0$) **M1**

After 8 complete years **A1**

Regular savings with compound interest

5 marks | Grade A

1. Work out the first year or two step by step.
2. Write the total as a series and identify a, r and n.
3. Use S_n, and round money only at the end.

Example. At the start of each year Tom pays £1000 into a savings account. Interest of 4% per year is added at the end of each year.
(a) Show that he has £2121.60 at the end of year 2.
(b) Find the amount in the account at the end of year 15, to the nearest pound.

Model answer.

(a) End of year 1: $1000 \times 1.04 = 1040$; end of year 2:

$$(1040 + 1000) \times 1.04 = 2121.60$$ **M1 A1**

(b) Total $= 1000(1.04) + 1000(1.04)^2 + \cdots + 1000(1.04)^{15}$: geometric with $a = 1040$, $r = 1.04$, $n = 15$ **M1**

$$S_{15} = \frac{1040(1.04^{15} - 1)}{1.04 - 1}$$ **M1**

= £20825 (to the nearest pound) **A1**

Sum to infinity in context, and a limitation

5 marks | Grade A*

1. Identify the geometric pattern and its first term.
2. Decide which distances are travelled once and which are travelled twice.
3. Use S_∞ because the heights keep shrinking.
4. For a limitation, refer to the context.

Example. A ball is dropped from a height of 5 m. After each bounce it rises to 60% of the height from which it fell.
(a) Find the total distance the ball travels before it comes to rest, according to this model.
(b) Give one limitation of the model.

Model answer.

(a) Heights after each bounce: 3, 1.8, 1.08, …: geometric with $a = 3$, $r = 0.6$ **M1**

Each of these heights is travelled twice (up and down): $2 \times \frac{3}{1 - 0.6} = 15$ **M1**

Total $= 5 + 15$ **M1** $= 20$ m **A1**

(b) The model predicts infinitely many bounces, but a real ball stops bouncing after a finite number of bounces (or: the ball may not rise to exactly 60% every time). **B1**

Don’t lose marks

- Using 0.15 instead of 0.85 as the multiplier for a 15% decrease.
- Off-by-one errors: the value after n years is ar^n when a is the starting value, not ar^{n-1}
- Using the nth term when the question asks for a total, or the other way round.
- Rounding money part way through, which makes the final answer wrong.
- Counting the first drop of a bouncing ball twice.

More tips

Memory tricks

- Multipliers: 4% up is × 1.04, 15% down is × 0.85, 2.5% up is × 1.025.
- Write out the first two or three values in a small table before you use a formula.
- Sense check: a depreciating value must stay positive, and a running total must increase as n increases.
- Turn decimal answers into the context: n = 7.88 means the value first drops below the target after 8 complete years.

Exam technique

- Answer in context: 'in week 20', 'after 8 complete years', '£20825'.
- For 'show that' money answers, write the unrounded value (e.g. 11054.25) before you round.
- When asked for a limitation or a refinement, link it to the situation described rather than writing something general.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Choose an arithmetic or geometric model**
A fixed amount added each time is arithmetic; a fixed percentage change each time is geometric.

Grade D — **Find a value in a given year**
Use the nth term, taking care over what n = 1 represents in the context.

Grade C — **Find a total over several periods**
Use S_n when the question asks for a total, e.g. the total distance run over 12 weeks.

Grade B — **Find when a model reaches a target**
Form an equation or inequality in n and solve it with a quadratic or with logarithms.

Grade A — **Model regular savings with compound interest**
Each payment earns interest for a different number of years, so the total is a geometric series.

Grade A* — **Criticise or refine a sequence model**
Explain in context why the model may fail and how it could be improved.

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