Geometric sequences and series

P4.5 · Pure mathematics · revision notes

In a geometric sequence you multiply by the same number, the common ratio, each time. You need the nth term, the sum of the first n terms (including its proof) and the sum to infinity of a convergent series. Questions often need logarithms to find n.

Key facts

- $u_n = ar^{n-1}$
- $S_n = \frac{a(1-r^n)}{1-r}$, $r \neq 1$
- $S_\infty = \frac{a}{1-r}$, for $|r| < 1$
- Convergent when $|r| < 1$, i.e. $-1 < r < 1$
- Consecutive terms p, q, s: $q^2 = ps$
- $\log(r^n) = n \log r$
- Dividing by $\log r$ when $0 < r < 1$ reverses an inequality

Notes

Geometric sequences

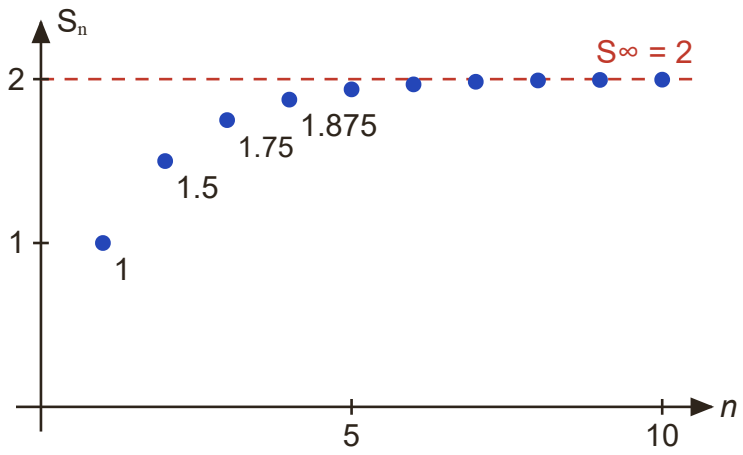
- Each term is the previous term multiplied by the **common ratio** r. Find r by dividing any term by the one before it.
- With first term a: $u_n = ar^{n-1}$. It is not in the formulae booklet, so learn it.
- If r is negative the terms alternate in sign; if $|r| < 1$ the terms get smaller in size.
- If p, q and s are consecutive terms, then $\frac{q}{p} = \frac{s}{q}$, so $q^2 = ps$

Sum of a finite geometric series

- $S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$, for $r \neq 1$. The second form is neater when r is greater than 1.
- Proof (learn this): $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. Multiply by r: $rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$
- Subtract: $S_n - rS_n = a - ar^n$, so $S_n(1-r) = a(1-r^n)$. Divide by $1-r$

Sum to infinity

- If $|r| < 1$, that is $-1 < r < 1$, then $r^n \to 0$ as $n \to \infty$, so $S_n \to \dfrac{a}{1-r}$. The series is **convergent**.
- $S_\infty = \dfrac{a}{1-r}$, valid only for $|r| < 1$



For 1 + ½ + ¼ + … (a = 1, r = ½) the sums get closer and closer to S∞ = 1 ÷ (1 − ½) = 2 but never pass it.

- If $|r| \geq 1$ the series is divergent and has no sum to infinity.
- If r involves x, e.g. $r = \dfrac{x}{3}$, the series converges when $\left|\dfrac{x}{3}\right| < 1$, that is $-3 < x < 3$

Finding n with logarithms

- A question such as 'find the least n for which $S_n > 1000$' leads to an inequality such as $1.2^n > 41$
- Take logs of both sides: $n \log 1.2 > \log 41$, so $n > \dfrac{\log 41}{\log 1.2}$
- If $0 < r < 1$, then $\log r < 0$, so dividing by it reverses the inequality.
- n must be a positive integer, so round the critical value to the correct side and check.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Find the ratio and first term, then the sum to infinity

5 marks | Grade B

1. Write each given term as an equation using $u_n = ar^{n-1}$
2. Divide the equations to eliminate a and solve for r.
3. Substitute back to find a.
4. Check that $|r| < 1$ before using $S_\infty = \dfrac{a}{1-r}$

Example. The second term of a geometric series is 48 and the fifth term is −6.
(a) Find the common ratio and the first term.
(b) Find the sum to infinity of the series.

Model answer.

(a) $ar = 48$ and $ar^4 = -6$, so $r^3 = \dfrac{-6}{48}$

$= -\dfrac{1}{8}$ **M1**

$r = -\dfrac{1}{2}$ **A1**; $a = 48 \div \left(-\dfrac{1}{2}\right) = -96$ **A1**

(b) $|r| < 1$, so $S_\infty = \dfrac{-96}{1-\left(-\frac{1}{2}\right)}$ **M1** $= -64$ **A1**

Find the least n using logarithms

4 marks | Grade A

1. Write S_n (or u_n) with the numbers substituted and set up the inequality.
2. Rearrange to get r^n on its own.
3. Take logs of both sides and divide by $\log r$, reversing the sign if $\log r < 0$
4. Give the least integer n and check the values either side.

Example. A geometric series has first term 5 and common ratio 1.2. Find the least value of n for which the sum of the first n terms exceeds 1000.

Model answer.

$\dfrac{5(1.2^n - 1)}{1.2 - 1} > 1000$ **M1**

$25(1.2^n - 1) > 1000 \Rightarrow 1.2^n > 41$ **A1**

$n > \dfrac{\log 41}{\log 1.2} = 20.37\ldots$ **M1**

$n = 21$ **A1**

Prove the sum formula

4 marks | Grade A

1. Write out S_n, including the last term ar^{n-1}
2. Multiply every term by r and write the result underneath, lining up equal terms.
3. Subtract: every term cancels except a and ar^n
4. Factorise and divide by $1-r$, noting that $r \neq 1$

Example. Prove that the sum of the first n terms of a geometric series with first term a and common ratio r, where $r \neq 1$, is $S_n = \dfrac{a(1-r^n)}{1-r}$

Model answer.

$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ — **B1**

$rS_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$ — **M1**

Subtracting: $S_n - rS_n = a - ar^n$ — **M1**

$S_n(1-r) = a(1-r^n)$, so $S_n = \dfrac{a(1-r^n)}{1-r}$ since $r \neq 1$ — **A1**

Combine a sum to infinity with other conditions

5 to 6 marks | Grade A*

1. Write each condition as an equation in a and r.
2. Eliminate a.
3. Solve for r and check that $|r| < 1$ so that the sum to infinity exists.
4. Find a for each valid value of r.

Example. A geometric series has sum to infinity 32. The sum of its first two terms is 24. Find the possible values of the common ratio and the corresponding first terms.

Model answer.

$\dfrac{a}{1-r} = 32 \Rightarrow a = 32(1-r)$ — **M1**

$a + ar = a(1+r) = 24$ — **M1**

$32(1-r)(1+r) = 24 \Rightarrow 1 - r^2 = \dfrac{3}{4}$ — **M1**

$r^2 = \dfrac{1}{4}$, so $r = \dfrac{1}{2}$ or $r = -\dfrac{1}{2}$; both satisfy $|r| < 1$ — **A1**

$r = \dfrac{1}{2}: a = 16$; $r = -\dfrac{1}{2}: a = 48$ — **A1**

Don't lose marks

- Using ar^n for the nth term instead of ar^{n-1}

- Forgetting the negative root of $r^2 = \frac{1}{4}$, or not checking that $|r| < 1$
- Quoting a sum to infinity when $|r| \geq 1$
- Not reversing the inequality after dividing by the log of a number between 0 and 1.
- Rounding the critical value down (20.37 to 20) when the least n is needed.
- Rounding r^n part way through, which can change the answer.

More tips

Memory tricks

- Divide term equations to cancel a: $\frac{ar^7}{ar^3} = r^4$
- Sense check: if a is positive and $0 < r < 1$, every partial sum is less than S_∞
- Use $\frac{a(r^n - 1)}{r - 1}$ when r is greater than 1 to avoid negative numbers.
- Check a 'least n' answer by working out S_{20} and S_{21} on your calculator.

Exam technique

- Say why S_∞ exists: 'since $|r| < 1$'.
- In log questions, show the line with the logs, e.g. $n > \frac{\log 41}{\log 1.2}$, before the decimal.
- The proof of S_n is bookwork: practise writing it out until you can do it quickly.
- Keep exact fractions for r and a wherever possible.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E

Find the nth term of a geometric sequence

Use $u_n = ar^{n-1}$, e.g. the 6th term of 3, 6, 12, … is $3 \times 2^5 = 96$

Grade D

Find the sum of a finite geometric series

Use $S_n = \frac{a(1 - r^n)}{1 - r}$ with the correct values of a, r and n.

Grade C

Find a and r from two terms

Divide one term equation by the other to eliminate a, e.g. $\frac{ar^4}{ar} = r^3$

Grade B

Find and use the sum to infinity

Check that $|r| < 1$, then use $S_\infty = \frac{a}{1 - r}$

Grade A

Use logarithms to find the least n

Rearrange to get r^n on its own, take logs, and take care with the direction of the inequality.

Grade A

Prove the sum of a geometric series

Write S_n, multiply by r, subtract and rearrange.

Grade A*

Solve problems combining sums, terms and convergence

Form equations from several conditions, solve for r and reject any value with $|r| \geq 1$ if a sum to infinity is involved.

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