Arithmetic sequences and series

P4.4 · Pure mathematics · revision notes

In an arithmetic sequence you add the same amount, the common difference, each time; an arithmetic series is the sum of its terms. You need the nth term and sum formulae, and you must be able to prove the sum formula. Questions range from finding the first term and common difference to solving for the number of terms.

Key facts

- $u_n = a + (n-1)d$
- $S_n = \frac{n}{2}(a + l)$, where l is the last term
- $u_n = S_n - S_{n-1}$
- Number of terms from a to l: $\frac{l - a}{d} + 1$
- $S_n = \frac{n}{2}[2a + (n-1)d]$
- Consecutive terms p, q, r: $2q = p + r$
- $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

Notes

Arithmetic sequences

- Each term is the previous term plus the **common difference** d, which can be negative or a fraction.
- With first term a: $u_n = a + (n-1)d$. It is not in the formulae booklet, so learn it.
- The last term of a series with n terms is $l = a + (n-1)d$
- If p, q and r are consecutive terms, then $q - p = r - q$, so $2q = p + r$
- To find how many terms are positive, solve $a + (n-1)d > 0$ and take the largest integer n that works.

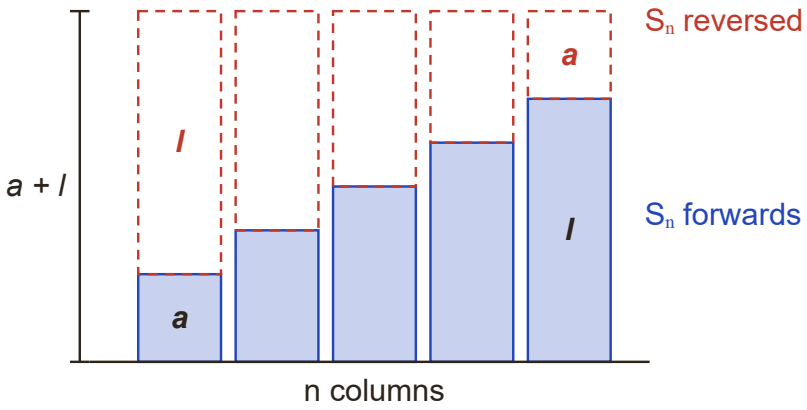
Arithmetic series

- An arithmetic series is the sum of the terms of an arithmetic sequence.
- $S_n = \frac{n}{2}[2a + (n-1)d]$ or $S_n = \frac{n}{2}(a + l)$. These are in the formulae booklet.
- The sum of the first n positive integers is $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$
- You can find a term from sums: $u_n = S_n - S_{n-1}$
- When you solve for n, n must be a positive integer. For a 'least n' problem, round the critical value up.

Proof of the sum formula (learn this)

$$S_n = a + (a+d) + (a+2d) + \ldots \\ +(a+(n-1)d)$$

- Write the sum in reverse: $S_n = (a+(n-1)d) + (a+(n-2)d) + \cdots + a$
- Add the two lines term by term. Each pair adds to $2a+(n-1)d$, and there are n pairs, so $2S_n = n[2a+(n-1)d]$



The series and its reverse fit together into n columns, each of height a + l, so $2S_n = n(a + l)$.

- Divide by 2: $S_n = \frac{n}{2}[2a+(n-1)d]$

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Find the first term and common difference 4 to 5 marks **Grade C**

1. Write each piece of information as an equation in a and d.
2. Simplify the sum equation by dividing out common factors.
3. Solve the two equations simultaneously.
4. Check your values with one of the original facts.

Example. The fifth term of an arithmetic series is 23 and the sum of the first 10 terms is 245. Find the first term and the common difference.

Model answer.

$a + 4d = 23$ **B1**

$\frac{10}{2}(2a+9d) = 245 \Rightarrow 2a+9d = 49$ **M1 A1**

Doubling the first equation gives $2a+8d=46$; subtracting gives $d=3$ **M1**

$a = 11$, $d = 3$ **A1**

Find the sum of terms that meet a condition

3 to 4 marks | Grade C

1. Find the first and last terms that fit the condition.
2. Count the terms carefully.
3. Use $S_n = \frac{n}{2}(a + l)$

Example. Find the sum of all the integers between 100 and 500 that are divisible by 7.

Model answer.

Smallest: $105 = 7 \times 15$; largest: $497 = 7 \times 71$ **B1**

Number of terms: $71 - 15 + 1 = 57$ **M1**

$$S = \frac{57}{2}(105 + 497)$$ **M1**

$$= 17157$$ **A1**

Find the least number of terms for a sum to exceed a value

4 marks | Grade B

1. Write S_n in terms of n.
2. Form the inequality and rearrange it into a quadratic.
3. Solve the quadratic equation to find the critical value.
4. Round up to the next integer and check the sums either side.

Example. An arithmetic series has first term 4 and common difference 3. Find the least number of terms needed for the sum to exceed 1200.

Model answer.

$$S_n = \frac{n}{2}[8 + 3(n - 1)]$$

$$= \frac{n}{2}(3n + 5)$$ **M1**

$$\frac{n}{2}(3n + 5) > 1200 \Rightarrow 3n^2 + 5n - 2400 > 0$$ **A1**

Critical value: $$n = \frac{-5 + \sqrt{25 + 28800}}{6}$$

$$= 27.46\ldots$$ **M1**

The least number of terms is 28. (Check: $S_{27} = 1161$, $S_{28} = 1246$.) **A1**

Prove the sum formula

4 marks **Grade A**

1. Write out S_n showing the first two and the last two terms.
2. Write the same sum in reverse order underneath.
3. Add the two lines: each pair makes $2a + (n-1)d$ and there are n pairs.
4. Divide by 2 and state the result.

Example. Prove that the sum of the first n terms of an arithmetic series with first term a and common difference d is $\frac{n}{2}[2a + (n-1)d]$

Model answer.

$S_n = a + (a+d) + \ldots$
$+(a + (n-2)d) + (a + (n-1)d)$ **B1**

S_n
$= (a + (n-1)d)$
$\quad + (a + (n-2)d)$
$+ \cdots + (a+d) + a$ **M1**

Adding the two lines, every pair of terms makes $2a + (n-1)d$ and there are n pairs, so
$2S_n = n[2a + (n-1)d]$ **M1**

So $S_n = \frac{n}{2}[2a + (n-1)d]$ **A1**

Don't lose marks

- Using nd instead of (n − 1)d in the nth term or the sum formula.
- Mixing up u_n (one term) and S_n (the total of n terms).
- Rounding a critical value such as 27.46 down to 27 when the question asks for the least n.
- Only halving part of the formula: the whole bracket $[2a + (n-1)d]$ is multiplied by $\frac{n}{2}$
- In the proof, leaving out the reversed line or not saying that there are n pairs.

More tips

Memory tricks

- Sum = number of terms × the mean of the first and last terms.
- Subtract two term equations to find d quickly: $u_9 - u_4 = 5d$
- Check a 'least n' answer by working out S_n for your n and for n − 1 on your calculator.
- The whole proof is 'forwards, backwards, add'.

Exam technique

- Label your equations (1) and (2) when you solve simultaneously.
- In a proof, set out each line clearly and finish with the formula; you cannot just quote it.
- For 'least n' questions, show the critical value and make your conclusion clear.
- Give n as a whole number and answer in context if there is one.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Find the nth term of an arithmetic sequence**
Use $u_n = a + (n-1)d$, e.g. the 20th term of 7, 11, 15, … is $7 + 19 \times 4 = 83$

Grade D — **Find the sum of an arithmetic series**
Use $S_n = \frac{n}{2}[2a + (n-1)d]$ or $S_n = \frac{n}{2}(a + l)$

Grade C — **Find a and d from given information**
Turn each fact into an equation in a and d, then solve the equations simultaneously.

Grade C — **Use consecutive terms to find an unknown**
If p, q and r are consecutive terms, then $q - p = r - q$

Grade B — **Find the number of terms from a sum**
Solve the quadratic in n from S_n, then choose the correct positive integer.

Grade A — **Prove the sum of an arithmetic series**
Write the series forwards and backwards, add the two lines, and divide by 2.

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