Sigma notation

P4.3 · Pure mathematics · revision notes

Sigma notation, $\sum$, is a short way of writing the sum of a series. You need to read it, write series in it and evaluate sums, usually by recognising an arithmetic or a geometric series. It often appears inside a longer question on sequences and series.

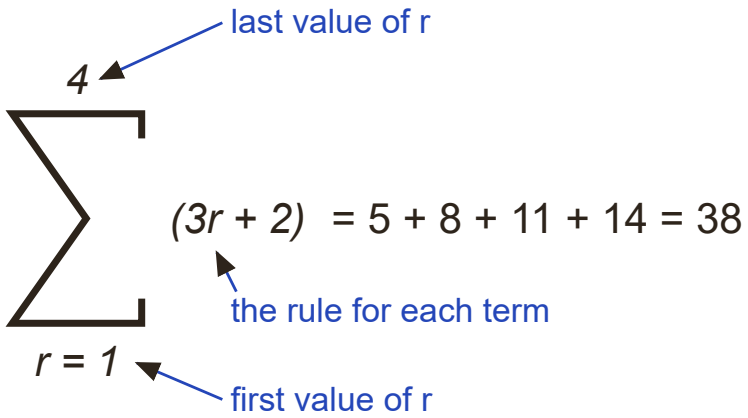
Key facts

- $\sum_{r=a}^{b} f(r)$ has $b - a + 1$ terms
- $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$
- Rule $A \times k^r$ → geometric with ratio k: $S_n = \frac{a(1-k^n)}{1-k}$, $S_\infty = \frac{a}{1-k}$ for $|k| < 1$
- $\sum_{r=k}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{k-1} f(r)$
- $\sum_{r=1}^{n} c = cn$
- Linear rule → arithmetic: $S_n = \frac{n}{2}(a+l) = \frac{n}{2}[2a + (n-1)d]$

Notes

Reading sigma notation

- $\sum_{r=1}^{n} f(r)$ means: substitute r = 1, 2, 3, …, n into f(r) and add the results.
- The number under $\sum$ is the first value of r; the number on top is the last value.



Substitute r = 1, 2, 3, 4 into the rule and add the results.

- From r = a to r = b there are $b - a + 1$ terms, e.g. $\sum_{r=5}^{20}$ has 16 terms.
- Adding a constant: $\sum_{r=1}^{n} c = cn$, e.g. $\sum_{r=1}^{10} 4 = 40$
- The letter r is only a counter: $\sum_{r=1}^{5} r$ and $\sum_{k=1}^{5} k$ are both 15.

Recognising the series

- If f(r) is linear, e.g. $5r - 2$, the terms form an **arithmetic** series. The common difference is the coefficient of r (here 5).
- If f(r) has the form $A \times k^r$, e.g. 3×2^r, the terms form a **geometric** series with common ratio k.
- Find the first term by substituting the bottom limit, not by reading off a coefficient: the first term of $\sum_{r=1}^{10} 3 \times 2^r$ is 6.
- $\sum_{r=1}^{n} r = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$, an arithmetic series.
- $\sum_{r=1}^{\infty}$ means the sum to infinity. It only exists for a geometric series whose common ratio k satisfies $|k| < 1$

Writing and rearranging sums

- To write a series in sigma notation, find a rule for the rth term and the limits: $3 + 7 + 11 + \cdots + 83 = \sum_{r=1}^{21}(4r - 1)$
- You can split a sum, $\sum(f(r) + g(r)) = \sum f(r) + \sum g(r)$, and take out constants, $\sum 3f(r) = 3\sum f(r)$
- Starting part way: $\sum_{r=k}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{k-1} f(r)$. Subtract up to k − 1, not up to k.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Evaluate an arithmetic series in sigma notation 3 marks **Grade C**

1. Substitute the bottom and top limits to get the first and last terms.
2. Count the terms: top limit − bottom limit + 1.
3. Use $S_n = \frac{n}{2}(a + l)$

Example. Find $\sum_{r=1}^{40}(3r + 2)$

Model answer.

First term (r = 1): 5; last term (r = 40): 122; 40 terms **M1**

$S_{40} = \frac{40}{2}(5 + 122)$ **M1**

$= 2540$ **A1**

Evaluate a geometric series in sigma notation

3 to 5 marks **Grade B**

1. Substitute the first value of r to get the first term a.
2. The common ratio is the number raised to the power r.
3. Use S_n for a finite sum, or S_∞ if the top limit is ∞ and the ratio is between −1 and 1.

Example. (a) Find $\sum_{r=1}^{10} 3 \times 2^r$.
(b) Find $\sum_{r=1}^{\infty} 12\left(\frac{2}{3}\right)^r$

Model answer.

(a) $a = 3 \times 2 = 6$, ratio 2, n = 10 **M1**

$S_{10} = \frac{6(2^{10} - 1)}{2 - 1}$ **M1** $= 6138$ **A1**

(b) $a = 12 \times \frac{2}{3} = 8$, ratio $\frac{2}{3}$ **M1**

$$S_\infty = \frac{8}{1 - \frac{2}{3}} = 24$$ **A1**

Use one sum to find another ('hence')

3 to 4 marks **Grade A**

1. Find the sum from r = 1 to the top limit.
2. Find the sum from r = 1 up to one less than the new bottom limit.
3. Subtract.
4. Check directly if you have time: here the first term is 21, the last is 81 and there are 16 terms, so the sum is $8 \times 102 = 816$

Example. (a) Find $\sum_{r=1}^{20}(4r + 1)$.
(b) Hence find $\sum_{r=5}^{20}(4r + 1)$

Model answer.

(a) $\frac{20}{2}(5 + 81) = 860$ **M1 A1**

(b) $$\sum_{r=1}^{4}(4r + 1) = 5 + 9 + 13 + 17 = 44$$ **M1**

$860 - 44 = 816$ **A1**

Find an unknown limit

4 marks | Grade A

1. Identify the series and its first term and common difference (or ratio).
2. Write the sum in terms of n.
3. Set it equal to the given value and solve (a quadratic for an arithmetic series, logarithms for a geometric one).
4. Reject any value that is not a positive integer.

Example. Given that $\sum_{r=1}^{n}(2r+3) = 396$, find the value of n.

Model answer.

Arithmetic series with $a = 5$, $d = 2$ **M1**

$$\frac{n}{2}[10 + 2(n-1)] = n(n+4)$$

$$= 396$$ **A1**

$n^2 + 4n - 396 = 0 \Rightarrow (n+22)(n-18) = 0$ **M1**

$n = 18$ (n must be a positive integer, so reject −22) **A1**

Don't lose marks

- Miscounting terms: r = 5 to r = 20 is 16 terms, not 15.
- Subtracting $\sum_{r=1}^{5}$ instead of $\sum_{r=1}^{4}$ when the sum starts at r = 5.
- Taking the first term of $\sum_{r=1}^{10} 3 \times 2^r$ as 3 instead of 6.
- Writing $\sum_{r=1}^{10} 4 = 4$ instead of 40.
- Confusing the counter r with the common ratio of a geometric series.

More tips

Memory tricks

- Always write out the first two terms and the last term before choosing a formula.
- Arithmetic sum = number of terms × (first term + last term) ÷ 2.
- Many calculators have a $\sum$ function: use it to check your answer, but still show your method.
- Quick check on counting: $\sum_{r=1}^{n}$ always has n terms.

Exam technique

- State the first term, the common difference or ratio, and the number of terms clearly: they earn method marks.
- If a part says 'hence', use your previous answer rather than starting again.
- If n must be a whole number, say why you reject a negative root.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E

Write out the terms of a sigma sum

Substitute each value of r from the bottom limit to the top limit, e.g. $\sum_{r=1}^{4} r^2 = 1 + 4 + 9 + 16$

Grade D

Count the terms and evaluate short sums

From r = a to r = b there are b − a + 1 terms; for a few terms, just add them.

Grade C

Evaluate an arithmetic series in sigma notation

A linear rule such as 3r + 2 gives an arithmetic series: find the first term, last term and number of terms, then use S_n

Grade B

Evaluate a geometric series in sigma notation

A rule such as 3×2^r gives a geometric series: find the first term by substituting, then use S_n or S_∞

Grade A

Change the limits of a sum

Use $\sum_{r=k}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{k-1} f(r)$, or find the first term, last term and number of terms directly.

Grade A

Find an unknown limit from a given sum

Write the sum in terms of n, set it equal to the value and solve, keeping only a positive integer n.

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