Sequences and recurrence relations

P4.2 · Pure mathematics · revision notes

A sequence can be given by a formula for its nth term or by a recurrence relation that generates each term from the one before. You need to generate terms, find unknown constants, and describe sequences as increasing, decreasing or periodic. These are usually short questions, often combined with sigma notation.

Key facts

- Increasing: $u_{n+1} > u_n$ for all n
- Periodic with order k: $u_{n+k} = u_n$ for all n, where k is the smallest such number
- $(-1)^n$: −1, 1, −1, 1, … (periodic, order 2)
- Decreasing: $u_{n+1} < u_n$ for all n
- Recurrence relation: $u_{n+1} = f(u_n)$, plus a first term
- Sum of N terms of a periodic sequence = complete cycles × cycle sum + leftover terms

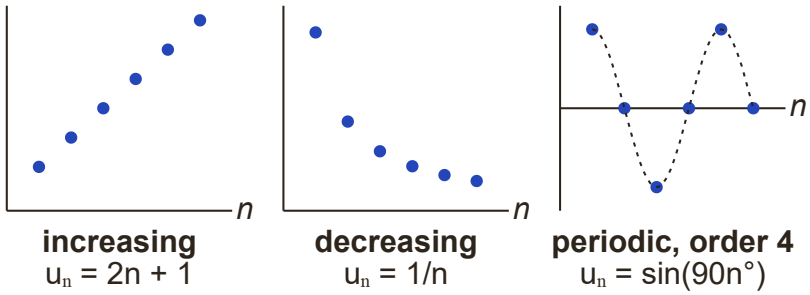
Notes

Two ways to define a sequence

- A sequence is an ordered list of terms $u_1, u_2, u_3, \ldots$. The subscript is the position of the term.
- An **nth term formula**, such as $u_n = 3n^2 - 1$, gives any term directly: $u_{10} = 299$
- A **recurrence relation**, such as $u_{n+1} = 2u_n - 3$ with $u_1 = 5$, gives each term from the one before. You must be told a first term.
- u_{n+1} means the term after u_n. It is not the same as $u_n + 1$
- Check whether a sequence starts at u_1 or at u_0

Increasing, decreasing and periodic

- **Increasing**: $u_{n+1} > u_n$ for all $n \in \mathbb{N}$, e.g. $u_n = 2n + 1$
- **Decreasing**: $u_{n+1} < u_n$ for all n, e.g. $u_n = \dfrac{1}{n}$
- **Periodic**: the terms repeat in a cycle, so $u_{n+k} = u_n$ for all n. The smallest such k is the **order** (or period).
- For example, $u_n = (-1)^n$ is periodic with order 2, and $u_n = \sin(90n°)$, which gives 1, 0, −1, 0, 1, …, is periodic with order 4.



increasing $u_n = 2n + 1$ | **decreasing** $u_n = 1/n$ | **periodic, order 4** $u_n = \sin(90n°)$

Plotting u_n against n shows the behaviour: always up, always down, or repeating in a cycle.

- Many sequences are none of these, e.g. $u_n = (-2)^n$ alternates in sign and grows in size.
- To prove that a sequence is increasing, show that $u_{n+1} - u_n > 0$ for all n. Checking a few terms is not a proof.

Working with recurrence relations

- To find unknown constants, write the terms you know in terms of the constants and solve the equations.
- Example: $u_{n+1} = au_n + b$ with $u_1 = 2$, $u_2 = 7$, $u_3 = 22$ gives $2a + b = 7$ and $7a + b = 22$, so $a = 3$ and $b = 1$
- To add many terms of a periodic sequence: (number of complete cycles) × (sum of one cycle), plus the leftover terms.
- If you get a quadratic for a constant, keep both roots unless the question rules one out.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Find terms and an unknown constant from a recurrence relation

4 to 6 marks | Grade C

1. Substitute n = 1 to get u_2, then n = 2 to get u_3, keeping the constant as a letter.
2. Expand brackets and simplify.
3. Set the expression equal to the given value and solve.
4. Keep every solution that the question allows.

Example. A sequence is defined by $u_1 = 3$, $u_{n+1} = ku_n - 4$, where k is a constant.
(a) Find u_3 in terms of k, giving your answer in its simplest form.
(b) Given that $u_3 = 11$, find the possible values of k.

Model answer.

(a) $u_2 = 3k - 4$ **B1**

$u_3 = k(3k - 4) - 4$ **M1**

$= 3k^2 - 4k - 4$ **A1**

(b) $3k^2 - 4k - 4 = 11 \Rightarrow 3k^2 - 4k - 15 = 0$ **M1**

$(3k + 5)(k - 3) = 0$ **M1**

$k = 3$ or $k = -\frac{5}{3}$ **A1**

Identify a periodic sequence and find a sum

5 to 6 marks | Grade A

1. Work out terms until one repeats the first term.
2. State that the sequence is periodic and give its order (the length of the cycle).
3. Add one complete cycle.
4. Divide the number of terms by the order: complete cycles × cycle sum, plus the leftover terms.

Example. A sequence is defined by $u_1 = 2$, $u_{n+1} = \frac{1}{1 - u_n}$.
(a) Find u_2, u_3 and u_4, and describe the sequence.
(b) Find $\sum_{r=1}^{100} u_r$

Model answer.

(a) $u_2 = \frac{1}{1-2} = -1$, $u_3 = \frac{1}{1-(-1)} = \frac{1}{2}$, $u_4 = \frac{1}{1-\frac{1}{2}} = 2$ **M1 A1**

The sequence is periodic with order 3 **B1**

(b) One cycle: $2 + (-1) + \frac{1}{2} = \frac{3}{2}$. 100 terms = 33 complete cycles + 1 term **M1**

$33 \times \frac{3}{2} + u_{100}$, where $u_{100} = u_1 = 2$ **M1**

$= 49.5 + 2 = 51.5$ **A1**

Prove that a sequence is increasing (or decreasing) 3 to 4 marks **Grade A**

1. Write u_{n+1} by replacing n with n + 1 everywhere.
2. Find $u_{n+1} - u_n$ and simplify it to a single expression.
3. Explain why it is positive (or negative) for every positive integer n.
4. Finish with a conclusion, e.g. 'so the sequence is increasing'.

Example. A sequence has nth term $u_n = \dfrac{2n+1}{n+1}$, $n \geq 1$. Prove that the sequence is increasing.

Model answer.

$$u_{n+1} - u_n = \frac{2n+3}{n+2} - \frac{2n+1}{n+1}$$ **M1**

$$= \frac{(2n+3)(n+1) - (2n+1)(n+2)}{(n+1)(n+2)}$$

$$= \frac{(2n^2+5n+3) - (2n^2+5n+2)}{(n+1)(n+2)}$$ **M1**

$$= \frac{1}{(n+1)(n+2)}$$ **A1**

For $n \geq 1$ the denominator is positive, so $u_{n+1} - u_n > 0$ and the sequence is increasing. **A1**

Don't lose marks

- Reading u_{n+1} as $u_n + 1$
- Checking a few terms and claiming the sequence is increasing 'for all n'. A proof needs a general argument.
- Miscounting the leftover terms in a periodic sum, e.g. using 34 cycles for 100 terms of order 3.
- Rejecting a negative or fractional value of a constant when the question allows it.
- Giving a multiple of the true order for a periodic sequence; the order is the smallest cycle length.

More tips

Memory tricks

- Calculator: type the first term and press =, then type the rule using Ans, e.g. 1 ÷ (1 − Ans), and keep pressing = to generate terms.
- Rewrite a fraction to see how it behaves: $\frac{2n+1}{n+1} = 2 - \frac{1}{n+1}$, and $\frac{1}{n+1}$ gets smaller as n grows, so the sequence increases.
- For a periodic sum, use the remainder: with order 3, $100 = 3 \times 33 + 1$, so $u_{100} = u_1$
- Write out the first five terms of any unfamiliar sequence; the pattern often jumps out.

Exam technique

- Show each substitution, e.g. $u_3 = k(3k - 4) - 4$, before you simplify.
- When asked to describe a sequence, use the exact words increasing, decreasing or periodic, and give the order of a periodic sequence.
- Keep exact fractions: rounding errors grow as you go through a recurrence relation.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Find terms from an nth term formula**
Substitute n = 1, 2, 3, … into a formula such as $u_n = 3n^2 - 1$

Grade D — **Generate terms from a recurrence relation**
Use a rule such as $u_{n+1} = 2u_n - 3$ with the first term to find $u_2, u_3, \ldots$ in turn.

Grade C — **Find unknown constants in a recurrence relation**
Write the terms in terms of the constant, form an equation from a given term and solve it.

Grade B — **Describe a sequence as increasing, decreasing or periodic**
Use the definitions: $u_{n+1} > u_n$ for all n, $u_{n+1} < u_n$ for all n, or terms repeating in a cycle.

Grade A — **Find the order and sum of periodic sequences**
Find the length of the cycle, then add many terms using complete cycles plus the leftover terms.

Grade A — **Prove that a sequence is increasing**
Show that $u_{n+1} - u_n > 0$ for every positive integer n, not just for a few terms.

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