Binomial expansion

P4.1 · Pure mathematics · revision notes

The binomial expansion lets you expand $(a+bx)^n$ without multiplying out brackets. For a positive integer n the expansion stops after n + 1 terms; for a negative or fractional n it is an infinite series that is only valid for some values of x, and it is used for approximations and with partial fractions. It comes up on both Paper 1 and Paper 2.

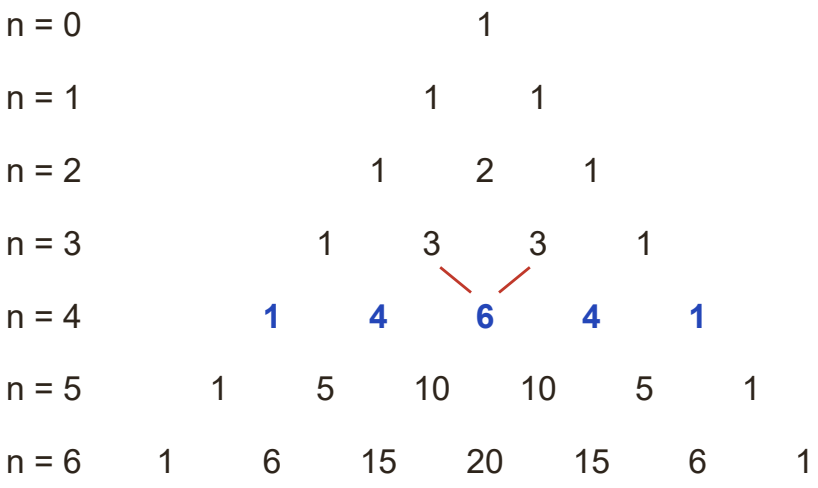
Key facts

- ${}^nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$
- General term: $\binom{n}{r}a^{n-r}b^r$
- $(a+bx)^n = a^n\left(1+\dfrac{b}{a}x\right)^n$, valid for $|x| < \left|\dfrac{a}{b}\right|$
- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$
- $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + b^n$, for positive integer n
- $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots$, valid for $|x| < 1$ when n is negative or a fraction

Notes

Positive integer powers

- $n! = n \times (n-1) \times \cdots \times 2 \times 1$, and $0! = 1$
- ${}^nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$. These are the numbers in row n of Pascal's triangle; your calculator has an nCr button.



Row 4 gives
$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Each number is the sum of the two above it (3 + 3 = 6). Row n gives the coefficients of $(a + b)^n$.

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n$$

- There are n + 1 terms. The power of a goes down by 1 each time and the power of b goes up by 1, so the powers in every term add up to n.
- When b is something like $-3x$, keep it in a bracket: $(-3x)^2 = 9x^2$ and $(-3x)^3 = -27x^3$
- The same coefficients give binomial probabilities: $P(X = r) = \binom{n}{r}p^r(1-p)^{n-r}$

Any rational power

- For any rational n:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

- If n is negative or a fraction, the series never stops, so you give the first few terms. It is only valid for $|x| < 1$
- To expand $(a+bx)^n$, take out a factor of a^n first: $(a+bx)^n = a^n\left(1+\frac{b}{a}x\right)^n$. Then use the formula with $\frac{b}{a}x$ in place of x.
- This expansion is valid for $\left|\frac{bx}{a}\right| < 1$, which means $|x| < \left|\frac{a}{b}\right|$
- Rewrite roots and fractions as powers first, e.g. $\frac{1}{\sqrt{4-x}} = (4-x)^{-\frac{1}{2}}$

Approximations and partial fractions

- When x is small the terms get small quickly, so the first few terms give a good approximation. The smaller |x| is, the better.
- To estimate a number, choose x so that the expression becomes that number, check that x is in the valid range, then substitute.
- For a fraction such as $\frac{3}{(1+x)(1-2x)}$, split it into partial fractions, write each part as a power, e.g. $2(1-2x)^{-1}$, expand each part and add.
- The combined expansion is only valid where every part is valid, so give the narrower range.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Find the first terms of $(a+bx)^n$ for a positive integer n

4 marks | Grade D

1. Write the pattern: a^n, then $\binom{n}{1}a^{n-1}(bx)$, $\binom{n}{2}a^{n-2}(bx)^2$, and so on.
2. Keep bx in a bracket so the number is raised to the power as well.
3. Work out each coefficient carefully and simplify.
4. Give the terms in ascending powers of x (constant first).

Example. Find the first four terms, in ascending powers of x, of the binomial expansion of $(2-3x)^6$. Give each term in its simplest form.

Model answer.

$2^6 = 64$ **B1**

$\binom{6}{1}2^5(-3x) + \binom{6}{2}2^4(-3x)^2 + \binom{6}{3}2^3(-3x)^3$ **M1**

$= -576x + 2160x^2 - 4320x^3$ **A1** (two terms correct) **A1** (all correct)

So $(2-3x)^6$
$= 64 - 576x + 2160x^2$
$- 4320x^3 + \ldots$

Find a coefficient in a product, or an unknown constant

5 marks (2 + 3) | Grade C

1. Only work out the terms you need.
2. For a product such as $(3+x)(2-3x)^6$, find every pair of terms that multiply to give the power you want.
3. For an unknown constant, write the coefficient in terms of it, set it equal to the given value and solve.
4. Keep both roots unless the question rules one out (e.g. 'k is positive').

Example. (a) Given that $(2-3x)^6 = 64 - 576x + 2160x^2 + \ldots$, find the coefficient of x^2 in the expansion of $(3+x)(2-3x)^6$.
(b) The coefficient of x^2 in the expansion of $(1+kx)^8$ is 252. Find the possible values of the constant k.

Model answer.

(a) $3 \times 2160 + 1 \times (-576)$ **M1**

$= 5904$ **A1**

(b) Term in x^2: $\binom{8}{2}(kx)^2 = 28k^2x^2$ **M1**

$28k^2 = 252 \Rightarrow k^2 = 9$ **M1**

$k = 3$ or $k = -3$ **A1**

Expand $(a+bx)^n$ for rational n and state the validity

4 to 6 marks | **Grade B**

1. Write the expression as a power, e.g. $(4-x)^{-\frac{1}{2}}$
2. Take out a^n: $4^{-\frac{1}{2}}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$
3. Substitute $\left(-\frac{x}{4}\right)$, in a bracket, for x in the $(1+x)^n$ formula.
4. Simplify each term, then multiply every term by a^n
5. Validity: $\left|\frac{bx}{a}\right| < 1$

Example. Find the first three terms, in ascending powers of x, of the binomial expansion of $\frac{1}{\sqrt{4-x}}$, giving each coefficient as a simplified fraction. State the range of values of x for which the expansion is valid.

Model answer.

$$(4-x)^{-\frac{1}{2}}$$

$$= 4^{-\frac{1}{2}}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}} \quad \textbf{B1}$$

$$\left(1-\frac{x}{4}\right)^{-\frac{1}{2}} = 1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)$$

$$+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2+\dots \quad \textbf{M1}$$

$$= 1+\frac{1}{8}x+\frac{3}{128}x^2+\dots \quad \textbf{A1}$$

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}+\frac{1}{16}x+\frac{3}{256}x^2+\dots \quad \textbf{A1}$$

Valid for $|x|<4$ **B1**

Use the expansion to find an approximation

2 to 3 marks **Grade A**

1. Find the value of x that turns the expression into the one in the question.
2. Check that this x is inside the valid range.
3. Substitute into your expansion, then rearrange if needed.
4. Give the answer to the accuracy asked for.

Example. Given that $\frac{1}{\sqrt{4-x}} = \frac{1}{2} + \frac{1}{16}x + \frac{3}{256}x^2 + \ldots$, substitute $x = 0.04$ to find an approximation for $\sqrt{11}$. Give your answer to 4 decimal places.

Model answer.

$x = 0.04$ is inside the valid range $|x| < 4$.

$$\frac{1}{\sqrt{3.96}} \approx \frac{1}{2} + \frac{0.04}{16} + \frac{3(0.04)^2}{256}$$
$$= 0.50251875$$ **M1**

$$\sqrt{3.96} = \sqrt{\frac{396}{100}} = \frac{6\sqrt{11}}{10} = \frac{3\sqrt{11}}{5}, \text{ so } \frac{5}{3\sqrt{11}} \approx 0.50251875$$ **M1**

$$\sqrt{11} \approx \frac{5}{3 \times 0.50251875}$$
$$= 3.3166 \quad \text{(4 d.p.)}$$ **A1**

Expand a fraction using partial fractions 6 to 8 marks **Grade A***

1. Split the fraction into partial fractions.
2. Write each part as a power, e.g. $\frac{2}{1-2x} = 2(1-2x)^{-1}$
3. Expand each part up to the term you need and add like terms.
4. Give the validity: the narrower of the ranges for the separate parts.

Example. $f(x) = \frac{3}{(1+x)(1-2x)}$

(a) Express f(x) in partial fractions.

(b) Hence find the series expansion of f(x), in ascending powers of x, up to and including the term in x^2.

(c) State the range of values of x for which your expansion is valid.

Model answer.

(a) $3 \equiv A(1-2x) + B(1+x)$ **M1**

$x = -1$: $3 = 3A$, so $A = 1$; $x = \frac{1}{2}$: $3 = \frac{3}{2}B$, so $B = 2$ **A1**

$f(x) = \frac{1}{1+x} + \frac{2}{1-2x}$

(b) $(1+x)^{-1} = 1 - x + x^2 + \ldots$ **M1**

$2(1-2x)^{-1}$

$= 2(1 + 2x + 4x^2 + \ldots)$

$= 2 + 4x + 8x^2 + \ldots$ **M1**

$f(x) = 3 + 3x + 9x^2 + \ldots$ **A1**

(c) Valid for $|x| < 1$ and $|2x| < 1$, so $|x| < \frac{1}{2}$ **B1**

Don't lose marks

- Leaving out the bracket round the second term: writing $-3x^2$ instead of $(-3x)^2 = 9x^2$
- Forgetting to raise a to the power when taking it out: $4^{-\frac{1}{2}} = \frac{1}{2}$, not 4 or 2.
- Sign slips with negative n or negative terms, e.g. $\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)$ is positive.
- Giving the validity of $(a+bx)^n$ as $|x| < 1$ when it should be $|x| < \left|\frac{a}{b}\right|$
- Substituting a value of x outside the valid range, which gives a meaningless approximation.
- With partial fractions, giving the wider validity range instead of the narrower one.

More tips

Memory tricks

- Quick check: put x = 0. Your expansion should give the value of the original expression at x = 0, e.g. $\frac{1}{\sqrt{4}} = \frac{1}{2}$
- Building the coefficients for any n: start with 1, then keep multiplying by the next factor on top (n, n − 1, n − 2, …) and dividing by the next whole number (1, 2, 3, …).
- Pascal's triangle is quicker than nCr for small n: row 4 is 1, 4, 6, 4, 1.
- $(1-x)^{-1} = 1 + x + x^2 + \ldots$ is a geometric series with common ratio x, which makes a handy check.
- The positive integer expansion is in the formulae booklet; the A level booklet also gives the $(1+x)^n$ series for rational n. Check the formula there rather than relying on memory.

Exam technique

- 'Ascending powers of x' means the constant term first, then x, then x^2, and so on.
- Show the unsimplified terms (binomial coefficients and brackets) before simplifying, so you earn the method mark even if you slip.
- Give coefficients as exact integers or fractions, not decimals, unless you are told otherwise.
- When a part says 'hence', use the expansion you have just found.
- State the validity when asked, either as $|x| < 4$ or as $-4 < x < 4$

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Expand $(a+b)^n$ for positive integer n**

Use Pascal's triangle or ${}^nC_r$ to write down all n + 1 terms, e.g. $(1+2x)^4 = 1 + 8x + 24x^2 + 32x^3 + 16x^4$

Grade D — **Find a given term or coefficient**

The term in x^r in $(a+bx)^n$ is $\binom{n}{r}a^{n-r}(bx)^r$, so you can find it without writing out the whole expansion.

Grade C — **Find an unknown constant from a coefficient**

Write the coefficient in terms of the constant, set it equal to the given value and solve, e.g. $28k^2 = 252$

Grade C — **Expand $(1+x)^n$ for negative or fractional n**

Use $1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$ to give the first few terms, valid for $|x| < 1$

Grade B — **Expand $(a+bx)^n$ and state its validity**

Take out a^n to get $a^n\left(1+\frac{b}{a}x\right)^n$, expand, and give the valid range $|x| < \left|\frac{a}{b}\right|$

Grade A — **Use an expansion to approximate a value**

Choose a value of x inside the valid range that turns the expression into the number you want, then substitute it.

Grade A* — **Expand a fraction using partial fractions first**

Split into partial fractions, expand each one, add the series, and give the narrower of the validity ranges.

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