Parametric equations in modelling

P3.4 · Pure mathematics · revision notes

Using parametric equations to model real situations, such as the path of a ball or a point on a turning wheel, with the parameter representing time or an angle. You find positions at given times, find when something happens (such as landing), find greatest heights and distances, convert to a Cartesian path, and comment on the model's assumptions and limitations.

Key facts

- Start position: $t = 0$
- $x = ut$: constant horizontal speed u
- Cartesian path: eliminate t
- Assumptions: particle, no air resistance, flat ground
- Hits the ground: $y = 0$ (reject $t = 0$ and negative t)
- Greatest height: vertex of $y = at^2 + bt + c$, at $t = -\dfrac{b}{2a}$
- Wheel: $x = r\sin(kt)$, $y = h - r\cos(kt)$; one revolution takes $\dfrac{2\pi}{k}$

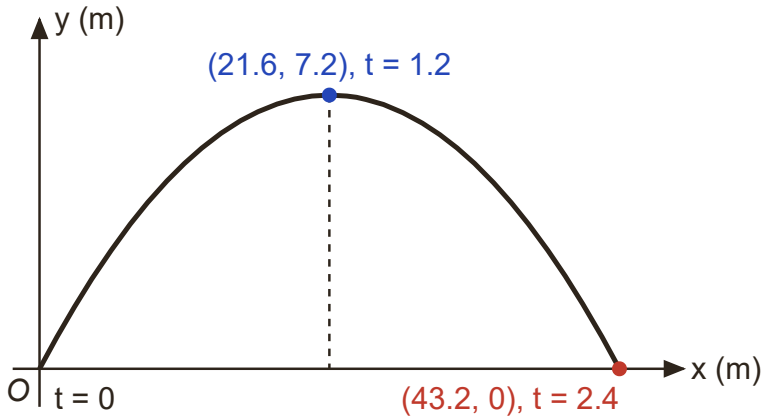
Notes

Setting up and using a model

- The parameter is often time t in seconds, with a restricted range, e.g. $0 \leq t \leq 2.4$ while a ball is in the air.
- Usually x is horizontal distance and y is height. A term like $18t$ in x means a constant horizontal speed of 18 m s^{-1}.
- Starting position: put $t = 0$. Position at a time: substitute t. Time at a position: solve one equation and check the other.

Projectile-type models

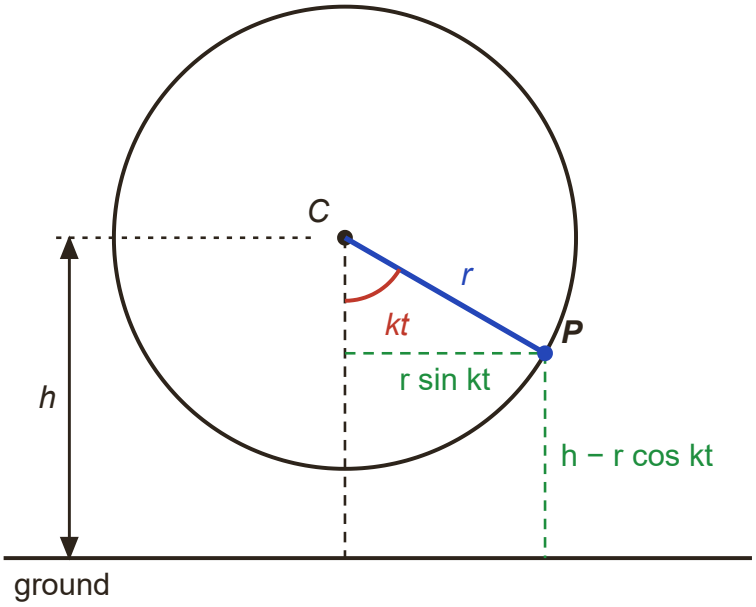
- Example: $x = 18t$, $y = 12t - 5t^2$ (metres, t in seconds).
- Lands when $y = 0$: $t(12 - 5t) = 0$, so $t = 2.4$ ($t = 0$ is the start). It lands $18 \times 2.4 = 43.2$ m away.
- Greatest height: $y = 7.2 - 5(t - 1.2)^2$, so 7.2 m when $t = 1.2$ (and $x = 21.6$ m).
- Cartesian path: $t = \dfrac{x}{18}$ gives $y = \dfrac{2x}{3} - \dfrac{5x^2}{324}$, for $0 \le x \le 43.2$: part of a parabola.



The path is part of a parabola: highest point at t = 1.2, back to the ground at t = 2.4.

Circular motion

- A point on a wheel of radius r, centre at height h, turning at a steady rate: $x = r\sin(kt)$, $y = h - r\cos(kt)$. It is at the lowest point when $t = 0$
- Eliminating t gives $x^2 + (y-h)^2 = r^2$: a circle with centre $(0, h)$. One revolution takes $\dfrac{2\pi}{k}$



P starts at the lowest point and turns through angle kt, so x = r sin kt and y = h − r cos kt.

- Use radians for trigonometric models.

Assumptions and limitations

- Typical assumptions: the object is a particle, there is no air resistance or wind, and the ground is horizontal.
- Refinements might include air resistance (the ball would not travel as far) or spin.
- Check answers make sense: t must lie in the stated range.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own, ft = follow through: you still earn the mark if you use your own earlier wrong answer correctly.

Projectile model: landing time, range and greatest height 5 marks **Grade C**

1. Set $y = 0$ and solve, rejecting $t = 0$
2. Substitute that time into x for the horizontal distance.
3. Find the vertex of $y(t)$ for the greatest height.

Example. A golf ball is hit from a point O on horizontal ground. Its position t seconds later is modelled by $x = 24t$, $y = 15t - 5t^2$, $0 \le t \le T$, where x m and y m are its horizontal and vertical distances from O.

(a) Find the value of T.

(b) Find the horizontal distance travelled by the ball.

(c) Find the greatest height of the ball.

Model answer.

(a) $15t - 5t^2 = 0$, so $5t(3 - t) = 0$ (M1)

$T = 3$ (A1)

(b) $24 \times 3 = 72$ m (B1 ft)

(c) Greatest height when $t = 1.5$ (M1)

$y = 22.5 - 11.25$

$= 11.25$ m (A1)

Find the Cartesian path and use it

4 marks **Grade B**

1. Rearrange $x(t)$ for t and substitute into $y(t)$
2. Simplify to the printed form.
3. Use the path (or the time) to test a condition in context.

Example. For the golf ball above:

(a) Show that its path has Cartesian equation $y = \dfrac{5x}{8} - \dfrac{5x^2}{576}$.

(b) A tree 4 m tall stands 66 m from O, in the path of the ball. Does the model predict that the ball passes over the tree? Give a reason.

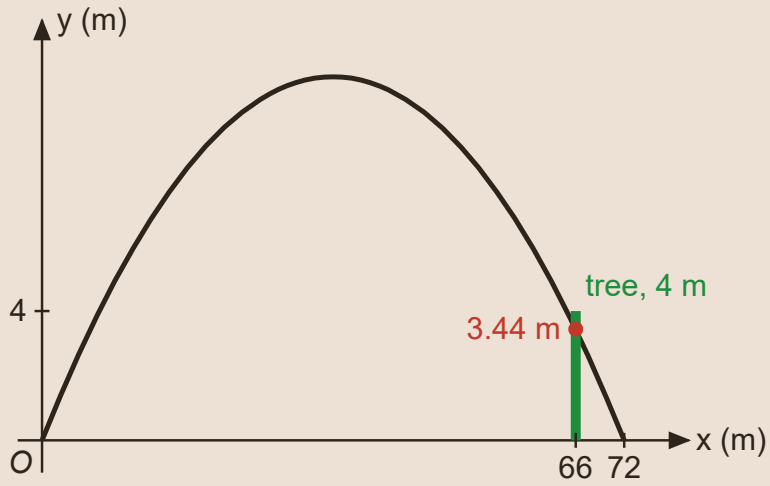
Model answer.

(a) $t = \dfrac{x}{24}$, so $y = 15\left(\dfrac{x}{24}\right) - 5\left(\dfrac{x}{24}\right)^2$ (M1)

$y = \dfrac{5x}{8} - \dfrac{5x^2}{576}$ (A1)

(b) At $x = 66$: $y = 41.25 - 37.8125$ (M1)

$= 3.44$ m (3 s.f.), which is less than 4 m, so the ball does not pass over the tree. (A1)



State limitations of the model

2 marks **Grade B**

1. Name a specific factor the model ignores.
2. Say how it would affect the prediction, in context.

Example. Give two limitations of the model for the golf ball.

Model answer.

The ball is modelled as a particle, so its size and spin are ignored. (B1)

Air resistance is ignored, so in reality the ball would not travel as far. (B1)

Model motion on a wheel 5 marks **Grade A**

1. Find the time for one revolution from the argument of the trig functions.
2. Use the greatest and least values of $\sin$ and $\cos$ for extreme positions.
3. Eliminate t with $\sin^2 + \cos^2 = 1$ and describe the path.

Example. A point P on a big wheel is modelled by $x = 12\sin\left(\frac{\pi t}{8}\right)$, $y = 14 - 12\cos\left(\frac{\pi t}{8}\right)$, where x m and y m are the horizontal and vertical positions of P relative to the point on the ground directly below the centre of the wheel, and t is the time in minutes.
(a) Find the time for one revolution.
(b) Find the greatest height of P above the ground.
(c) Find a Cartesian equation for the path of P and describe it.

Model answer.

(a) $\frac{2\pi}{\pi/8} = 16$ minutes (B1)

(b) Greatest when $\cos\left(\frac{\pi t}{8}\right) = -1$: $14 + 12 = 26$ m (B1)

(c) $\sin\left(\frac{\pi t}{8}\right) = \frac{x}{12}$ and $\cos\left(\frac{\pi t}{8}\right) = \frac{14 - y}{12}$, then use $\sin^2 + \cos^2 = 1$ (M1)

$x^2 + (y - 14)^2 = 144$ (A1)

A circle with centre $(0, 14)$ and radius 12 m (B1)

Don’t lose marks

- Giving $t = 0$ as the landing time.
- Leaving the answer as a value of t when a distance or position is asked for.
- Using degrees instead of radians in trigonometric models.
- Ignoring the parameter's range when stating the domain of the Cartesian path.

More tips

Memory tricks

- For $y = bt - ct^2$ starting from ground level, the greatest height is at half the flight time.
- Units: with t in seconds and x in metres, the coefficient of t in $x = ut$ is in m s^{-1}.
- A quick sketch of the path shows which root, or which part of the curve, you need.

Exam technique

- Interpret answers in context: 'The ball lands 72 m from O after 3 seconds.'
- For limitations, name a specific factor (air resistance, wind, spin, uneven ground) and say what effect it has.
- Round only final answers, give units, and use 3 significant figures unless told otherwise.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade D — **Find positions at given times**
Substitute the value of t into both equations and give the answers in context with units.

Grade C — **Find when a condition is met**
e.g. set $y = 0$ to find when a ball lands, and choose the root that fits the context.

Grade C — **Interpret constants and the parameter**
e.g. the coefficient of t in $x = 18t$ is a constant horizontal speed of 18 m s^{-1}.

Grade B — **Find greatest heights and distances**
Find the vertex of $y(t)$ by completing the square or by symmetry, then find the position.

Grade B — **Convert a model to a Cartesian path**
Eliminate t to get y in terms of x, with a domain that matches the context.

Grade A — **Model circular motion with trigonometric parameters**
e.g. $x = r\sin(kt)$, $y = h - r\cos(kt)$ for a point on a wheel turning at a steady rate.

Grade A — **Criticise and refine a parametric model**
e.g. air resistance, spin and wind are ignored, so the real path is not exactly a parabola.

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