Parametric equations

P3.3 · Pure mathematics · revision notes

Curves described by giving x and y separately in terms of a parameter, usually t or θ. You convert between parametric and Cartesian forms by eliminating the parameter, find points and intersections, and use the range of the parameter to give the domain and range of the curve.

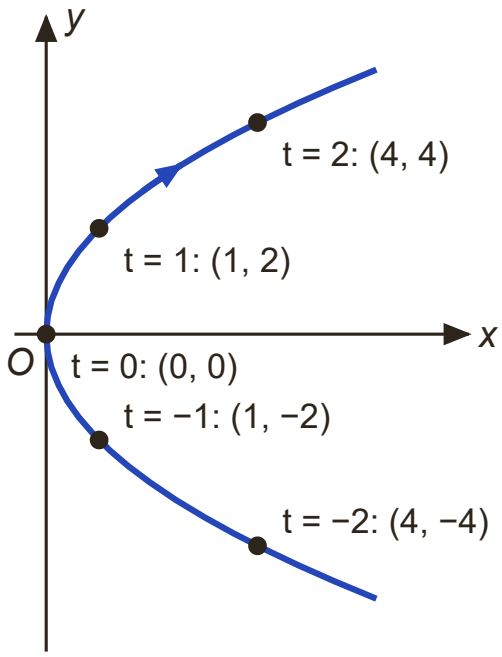
Key facts

- Point: substitute t into both x and y
- $\sin^2 t + \cos^2 t = 1$
- $\sin 2t = 2\sin t \cos t$
- $x = a + r\cos t$, $y = b + r\sin t$: circle, centre (a, b), radius r
- Eliminate t: rearrange one equation for t and substitute
- $\cos 2t = 1 - 2\sin^2 t = 2\cos^2 t - 1$
- $1 + \tan^2 t = \sec^2 t$
- Domain of the curve = the x values given by the range of t

Notes

What parametric equations are

- $x = f(t)$, $y = g(t)$: each value of the parameter t gives one point (x, y) on the curve.
- To find a point, substitute t. To find t at a given point, solve one equation and check the value in the other.



x = t², y = 2t: each value of t gives one point. The arrow shows the direction of increasing t.

- The range of t decides which part of the curve you get. The set of x values it gives is the domain of the Cartesian equation, and the set of y values is its range.

Converting to Cartesian form

- Algebraic: rearrange the simpler equation for t and substitute. E.g. $x = 2t - 1$, $y = \dfrac{3}{t}$ gives $t = \dfrac{x+1}{2}$, so $y = \dfrac{6}{x+1}$
- Trigonometric: make $\sin t$ and $\cos t$ the subjects and use $\sin^2 t + \cos^2 t = 1$. E.g. $x = 3 + 2\cos t$, $y = -1 + 2\sin t$ gives $(x-3)^2 + (y+1)^2 = 4$: a circle with centre $(3, -1)$ and radius 2.
- Double angles: $x = \sin t$, $y = \cos 2t$ gives $y = 1 - 2x^2$, with $-1 \leq x \leq 1$
- Other identities: $x = \tan t$, $y = \sec t$ gives $y^2 = 1 + x^2$, using $1 + \tan^2 t = \sec^2 t$

Intersections, domain and range

- The curve meets the x-axis where $y(t) = 0$ and the y-axis where $x(t) = 0$; find t, then the coordinates.
- For the line $ax + by = c$, substitute $x(t)$ and $y(t)$ to get an equation in t
- Domain and range: $x = t^2 + 1$, $t \in \mathbb{R}$, gives $x \geq 1$; $x = 3\cos t$ gives $-3 \leq x \leq 3$. Give the Cartesian equation together with its domain.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Find a Cartesian equation (algebraic)

3 marks | Grade C

1. Rearrange the linear equation for t
2. Substitute into the other equation.
3. Simplify into the form asked for.

Example. A curve has parametric equations $x = 2t + 1$, $y = t^2 - 4t$, $t \in \mathbb{R}$. Show that a Cartesian equation of the curve is $y = \dfrac{(x-1)(x-9)}{4}$

Model answer.

$t = \dfrac{x-1}{2}$, substituted into y (M1)

$y = \dfrac{(x-1)^2}{4} - 2(x-1)$ (A1)

$y = \dfrac{(x-1)\left[(x-1)-8\right]}{4}$

$= \dfrac{(x-1)(x-9)}{4}$ (A1)

Convert trigonometric parametric equations and give the domain

5 marks | Grade B

1. Choose an identity that links the two trigonometric functions.
2. Substitute $\sin t$ or $\cos t$ in terms of x or y
3. Use the range of t to find the domain and range.

Example. A curve has parametric equations $x = 4\sin t$, $y = 3\cos 2t$, $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$.

(a) Find a Cartesian equation of the curve in the form $y = f(x)$.

(b) State the domain and range of f

Model answer.

(a) $\cos 2t = 1 - 2\sin^2 t$ (M1)

$\sin t = \dfrac{x}{4}$, so $y = 3\left(1 - 2 \times \dfrac{x^2}{16}\right)$ (M1)

$y = 3 - \dfrac{3}{8}x^2$ (A1)

(b) Domain $-4 \le x \le 4$ (B1)

Range $-3 \le f(x) \le 3$ (B1)

Find where a parametric curve meets a line

4 marks | **Grade C**

1. Substitute $x(t)$ and $y(t)$ into the line's equation.
2. Solve the equation for t
3. Substitute each t back to find the coordinates.

Example. The curve C has parametric equations $x = t^2$, $y = 2t$. Find the coordinates of the points where C meets the line $y = x - 3$

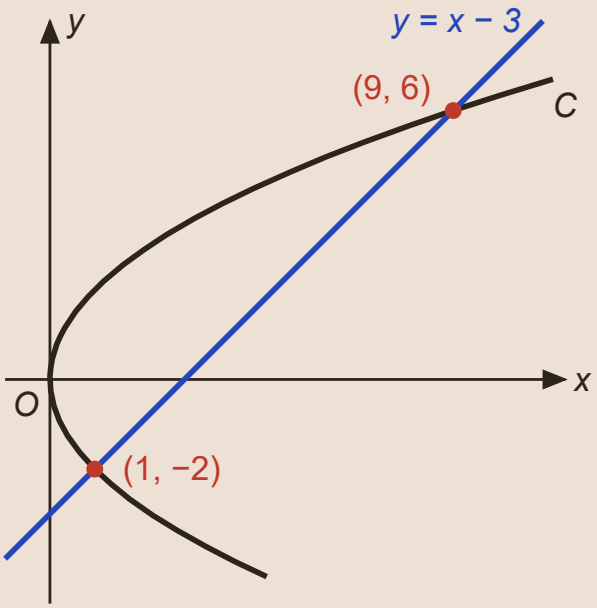
Model answer.

$2t = t^2 - 3$ (M1)

$(t-3)(t+1) = 0$, so $t = 3$ or $t = -1$ (A1)

Substituting into $x = t^2$, $y = 2t$ (M1)

$(9, 6)$ and $(1, -2)$ (A1)



Don't lose marks

- Leaving out the domain when the parameter is restricted, e.g. $x = \sqrt{t}$ gives $x \geq 0$
- Squaring wrongly: $\left(\frac{x}{4}\right)^2 = \frac{x^2}{16}$, not $\frac{x^2}{4}$
- Giving values of t instead of coordinates when asked for points.
- Using $\cos 2t = 1 - \sin^2 t$; the correct identity is $\cos 2t = 1 - 2\sin^2 t$

More tips

Memory tricks

- Rearrange whichever equation is linear in t: it avoids $\pm$ square roots.
- Check your Cartesian equation with one value of t: the point it gives must satisfy it.
- Same angle in $\sin$ and $\cos$: use $\sin^2 + \cos^2 = 1$. Angles t and $2t$: use a double angle formula.
- If you cannot recognise the curve, make a table of a few t values and plot the points.

Exam technique

- If the question asks for the form $y = f(x)$, make y the subject and simplify fully.
- Link the domain of f to the range of the parameter, using $\leq$ where end points are included.
- In 'show that' conversions, write down the identity you use before substituting.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **Find a point from a parameter value**
Substitute the value of t into both x and y

Grade D — **Convert to Cartesian form by substitution**
Rearrange one equation for t and substitute: $x = t + 1$, $y = t^2$ gives $y = (x - 1)^2$

Grade C — **Find where a curve meets an axis or line**
Set $y = 0$ or $x = 0$, or substitute $x(t)$ and $y(t)$ into the line's equation, then solve for t

Grade C — **Convert trigonometric parametric equations**
Use identities such as $\sin^2 t + \cos^2 t = 1$ or $\cos 2t = 1 - 2\sin^2 t$ to eliminate t

Grade B — **State the domain and range of the curve**
Use the range of the parameter: $x = 2\cos t$ gives $-2 \leq x \leq 2$

Grade B — **Sketch a curve given parametrically**
Recognise the Cartesian form (e.g. a circle) or plot points, and draw only the part given by the parameter.

Grade A — **Convert harder parametric equations with domains**
e.g. using $1 + \tan^2 t = \sec^2 t$, or with a restricted parameter, and state $y = f(x)$ with its domain.

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