Circles

P3.2 · Pure mathematics · revision notes

The equation of a circle $(x-a)^2+(y-b)^2=r^2$, finding the centre and radius by completing the square, and using circle properties: the angle in a semicircle is a right angle, the perpendicular from the centre to a chord bisects the chord, and a tangent is perpendicular to the radius. Circle questions are often long (8 to 10 marks) and combine straight lines, simultaneous equations and the discriminant.

Key facts

- $(x-a)^2+(y-b)^2=r^2$: centre (a,b), radius r
- Tangent $\perp$ radius at the point of contact
- Angle in a semicircle $=90°$
- Point inside the circle if its distance from the centre is less than r
- $x^2+y^2+2gx+2fy+c=0$: centre $(-g,-f)$, radius $\sqrt{g^2+f^2-c}$
- Perpendicular from the centre bisects a chord
- Line and circle: substitute, then b^2-4ac: >0 two points, $=0$ tangent, <0 none

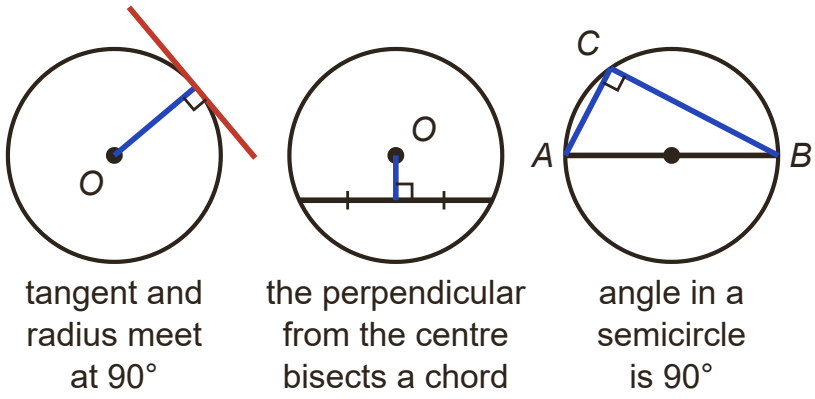
Notes

The equation of a circle

- Centre (a,b), radius r: $(x-a)^2+(y-b)^2=r^2$. Centre the origin: $x^2+y^2=r^2$
- Watch the signs: $(x+4)^2+(y-1)^2=9$ has centre $(-4,1)$ and radius 3.
- From the expanded form, complete the square for x and for y: $x^2+y^2+8x-2y+8=0$ gives $(x+4)^2-16+(y-1)^2-1+8=0$, so $(x+4)^2+(y-1)^2=9$
- The right-hand side must be positive for a real circle, which gives conditions on unknown constants: $x^2+y^2-4x+k=0$ is $(x-2)^2+y^2=4-k$, so $k<4$
- A point is inside, on or outside the circle if its distance from the centre is less than, equal to or greater than r

Circle properties

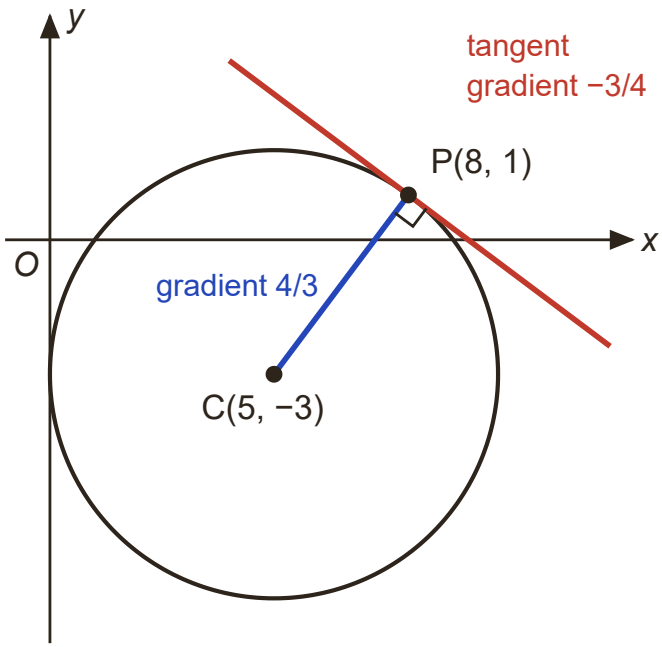
- A tangent at P is perpendicular to the radius at P
- The perpendicular from the centre to a chord bisects the chord, so the perpendicular bisector of any chord passes through the centre.
- The angle in a semicircle is a right angle: if AB is a diameter and C is on the circle, angle $ACB = 90°$. Conversely, if angle $ACB = 90°$, then AB is a diameter of the circle through A, B and C



The three circle facts you use most in coordinate geometry.

Lines and circles

- Tangent at P: find the gradient of the radius CP, where C is the centre, take the negative reciprocal, and use $y - y_1 = m(x - x_1)$ through P



Radius gradient 4/3, so the tangent gradient is −3/4 (negative reciprocal).

- Intersections: substitute the line into the circle's equation and solve the quadratic.
- The discriminant of that quadratic: > 0 two points (a chord), $= 0$ one point (a tangent), < 0 no points.
- Circle through three points: find where the perpendicular bisectors of two chords meet; that is the centre.
- Length of a tangent from an outside point T to the circle: $\sqrt{CT^2 - r^2}$, by Pythagoras.

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Find the centre and radius 3 marks **Grade C**

1. Group the x terms and the y terms.
2. Complete the square for each and move the constants to the right.
3. Read off the centre (change the signs) and square root for the radius.

Example. The circle C has equation $x^2 + y^2 - 10x + 6y + 9 = 0$. Find the coordinates of the centre of C and the radius of C

Model answer.

$(x-5)^2 - 25 + (y+3)^2 - 9 + 9 = 0$ (M1)

Centre $(5, -3)$ (A1)

$(x-5)^2 + (y+3)^2 = 25$, so radius 5 (A1)

Find the equation of a tangent to a circle 4 marks **Grade C**

1. Find the gradient of the radius from the centre to the point.
2. Take the negative reciprocal for the tangent's gradient.
3. Use $y - y_1 = m(x - x_1)$ through the point.

Example. The point $P(8, 1)$ lies on the circle $(x-5)^2 + (y+3)^2 = 25$. Find an equation of the tangent to the circle at P, giving your answer in the form $ax + by + c = 0$

Model answer.

Gradient of radius $= \dfrac{1-(-3)}{8-5}$

$= \dfrac{4}{3}$ (M1)

Tangent gradient $-\dfrac{3}{4}$ (M1)

$y - 1 = -\dfrac{3}{4}(x-8)$ (M1)

$3x + 4y - 28 = 0$ (A1)

Use a diameter and the angle in a semicircle

5 marks **Grade B**

1. The centre is the midpoint of the diameter.
2. The radius is half the length of the diameter.
3. Use gradients to show a right angle, or quote the angle in a semicircle.

Example. The points $A(-1, 2)$ and $B(7, 8)$ are the ends of a diameter of the circle C.
(a) Find an equation for C.
(b) The point $D(0, 9)$ lies on C. Show that angle $ADB = 90°$

Model answer.

(a) Centre $(3, 5)$ (B1)

$AB = \sqrt{8^2 + 6^2} = 10$, so $r = 5$ (M1)

$(x - 3)^2 + (y - 5)^2 = 25$ (A1)

(b) Gradient $AD = 7$ and gradient $BD = -\frac{1}{7}$ (M1)

$7 \times \left(-\frac{1}{7}\right) = -1$, so AD is perpendicular to BD and angle $ADB = 90°$. (A1)

Find when a line is a tangent (discriminant)

5 marks **Grade A**

1. Substitute the line into the circle's equation.
2. Collect into a quadratic in x with coefficients involving the unknown.
3. Set $b^2 - 4ac = 0$ and solve.

Example. The line $y = kx$ is a tangent to the circle $(x - 4)^2 + y^2 = 8$. Find the possible values of k

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Model answer.

$(x - 4)^2 + k^2x^2 = 8$ (M1)

$(1 + k^2)x^2 - 8x + 8 = 0$ (A1)

Tangent: $b^2 - 4ac = 0$ (M1)

$64 - 32(1 + k^2) = 0$ (A1)

$k = 1$ or $k = -1$ (A1)

Don't lose marks

- Giving r^2 as the radius, e.g. radius 25 instead of 5.
- Sign errors in the centre from the completed square.
- Forgetting to subtract the squared constants when completing the square.

- ✗ Using the radius gradient for the tangent instead of its negative reciprocal.
- ✗ Squaring wrongly: $(kx)^2 = k^2x^2$, not kx^2

More tips

Memory tricks

- The centre's signs flip: $(x + 4)^2$ means the x-coordinate of the centre is -4
- If you get a negative value for r^2, recheck your completing the square.
- Check a tangent: its gradient times the radius gradient should be -1
- Sketch the circle with its centre: it shows which property (tangent, chord or semicircle) to use.

Exam technique

- Draw a sketch with the centre, radius and any tangent or chord, and mark right angles.
- To show a point lies on a circle, substitute and show the left-hand side equals r^2 exactly, then conclude.
- Name the property you use, e.g. 'the tangent is perpendicular to the radius'.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E — **State the centre and radius from the equation**
$(x - a)^2 + (y - b)^2 = r^2$ has centre (a, b) and radius r

Grade D — **Write the equation from centre and radius**
Substitute into $(x - a)^2 + (y - b)^2 = r^2$; find r with the distance formula if needed.

Grade C — **Complete the square to find centre and radius**
e.g. $x^2 + y^2 - 6x + 4y - 12 = 0$ becomes $(x - 3)^2 + (y + 2)^2 = 25$: centre $(3, -2)$, radius 5.

Grade C — **Find the equation of a tangent**
The tangent is perpendicular to the radius at that point, so use the negative reciprocal of the radius gradient.

Grade B — **Use chord and semicircle properties**
The perpendicular bisector of a chord passes through the centre; an angle in a semicircle is $90°$

Grade B — **Find where a line meets a circle**
Substitute the line into the circle's equation and solve the resulting quadratic.

Grade A — **Use the discriminant for tangency conditions**
The line is a tangent when the substituted quadratic has $b^2 - 4ac = 0$, and misses when $b^2 - 4ac < 0$

Grade A — **Find the circle through three points**
The centre is where the perpendicular bisectors of two chords meet.

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