Transformations of graphs

P2.9 · Pure mathematics · revision notes

How the graph of $y = f(x)$ changes under translations, stretches and reflections, and combinations of these, together with the modulus graphs $y = |f(x)|$ and $y = f(|x|)$. You sketch transformed graphs from a given sketch, showing where turning points, intercepts and asymptotes move to, typically for 3 to 6 marks.

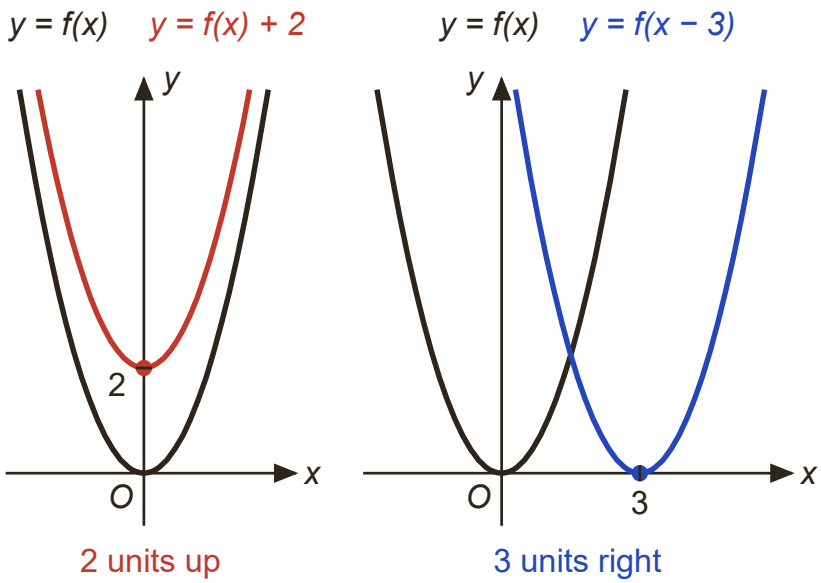
Key facts

- $f(x) + a$: translation $\begin{pmatrix} 0 \\ a \end{pmatrix}$
- $af(x)$: stretch parallel to the y-axis, scale factor a
- $-f(x)$: reflect in the x-axis; $f(-x)$: reflect in the y-axis
- $f(|x|)$: reflect the $x \geq 0$ part in the y-axis
- (p, q) under $y = f(ax + b)$ $\rightarrow \left(\dfrac{p - b}{a}, q\right)$
- $f(x + a)$: translation $\begin{pmatrix} -a \\ 0 \end{pmatrix}$
- $f(ax)$: stretch parallel to the x-axis, scale factor $\dfrac{1}{a}$
- $|f(x)|$: reflect the parts below the x-axis in the x-axis
- (p, q) under $y = af(x) + b \rightarrow (p, aq + b)$

Notes

Translations

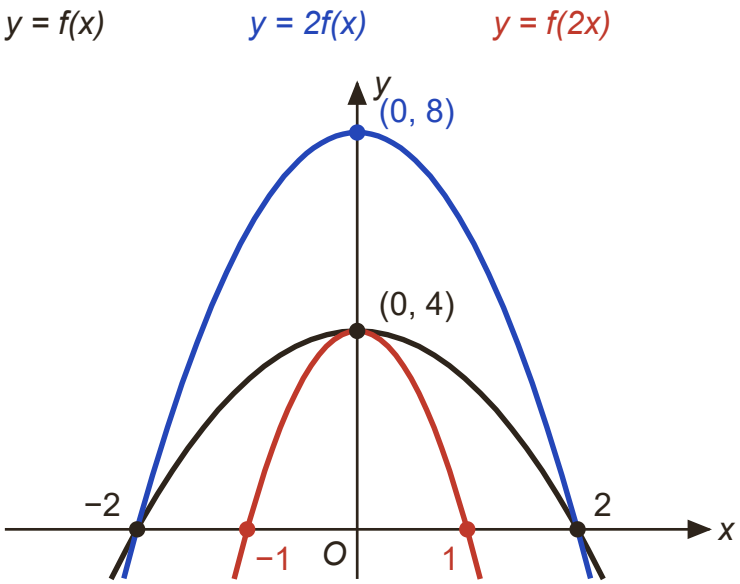
- $y = f(x) + a$: translation by $\begin{pmatrix} 0 \\ a \end{pmatrix}$, i.e. a units up. A horizontal asymptote $y = k$ becomes $y = k + a$
- $y = f(x + a)$: translation by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$, i.e. a units left. So $f(x - 3)$ moves the graph 3 units right. A vertical asymptote $x = k$ becomes $x = k - a$



With f(x) = x²: f(x) + 2 moves the curve 2 up; f(x − 3) moves it 3 to the right (the opposite of the sign).

Stretches and reflections

- $y = af(x)$: stretch parallel to the y-axis, scale factor a. Every y-coordinate is multiplied by a; points on the x-axis stay put.
- $y = f(ax)$: stretch parallel to the x-axis, scale factor $\frac{1}{a}$. Every x-coordinate is divided by a; points on the y-axis stay put.



With f(x) = 4 − x²: 2f(x) doubles every y-coordinate; f(2x) halves every x-coordinate.

- $y = -f(x)$: reflection in the x-axis. $y = f(-x)$: reflection in the y-axis.
- Rule of thumb: changes outside f affect y and do what you expect; changes inside the bracket affect x and do the opposite.

Modulus graphs

- $y = |f(x)|$: reflect any part of the graph below the x-axis in the x-axis. The graph never goes below the x-axis.
- $y = f(|x|)$: keep the part for $x \geq 0$, remove the part for $x < 0$, and reflect the kept part in the y-axis. The result is symmetric about the y-axis.

Combinations

- Do one transformation at a time, tracking key points: turning points, intercepts and asymptotes.
- Vertical: for $y = af(x) + b$, stretch first, then translate: $(p, q) \to (p, aq + b)$
- Horizontal: for $y = f(ax + b)$, $(p, q) \to \left(\frac{p - b}{a}, q\right)$
- Example: under $y = 3f(x + 2) - 1$, the point $(4, 5)$ moves to $(2, 14)$

How to answer each type of question

M = method mark, A = answer mark, B = a mark on its own.

Sketch a transformed graph from a given one

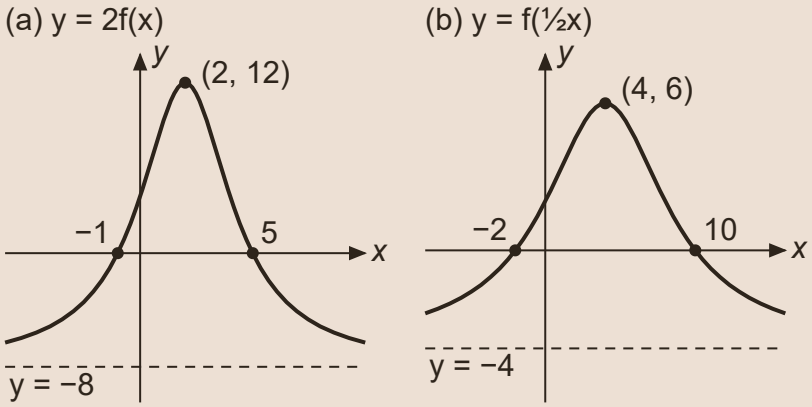
4 marks | Grade C

1. Identify the transformation for each part.
2. Move each key point and each asymptote.
3. Sketch the same basic shape and label the new coordinates and asymptote equations.

Example. The curve $y = f(x)$ has a maximum point at $(2, 6)$, crosses the x-axis at $(-1, 0)$ and $(5, 0)$, and has the horizontal asymptote $y = -4$.

On separate diagrams, sketch (a) $y = 2f(x)$ (b) $y = f\left(\frac{1}{2}x\right)$, showing the maximum point, the points where the curve crosses the x-axis and the asymptote.

Model answer.



(a) Same shape with maximum $(2, 12)$ (B1)

Crosses at $(-1, 0)$ and $(5, 0)$, asymptote $y = -8$ (B1)

(b) Same shape with maximum $(4, 6)$ (B1)

Crosses at $(-2, 0)$ and $(10, 0)$, asymptote $y = -4$ (B1)

Describe a transformation fully

2 marks | Grade C

1. Compare the new equation with $y = f(x)$ to see what has changed.
2. Name the transformation.
3. Give its vector, or its direction and scale factor, or its mirror line.

Example. Describe fully the single transformation that maps the curve $y = \frac{1}{x}$ onto the curve $y = \frac{1}{x + 2} - 5$

Model answer.

Translation (B1)

by the vector $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$ (B1)

Sketch y = |f(x)| or y = f(|x|)

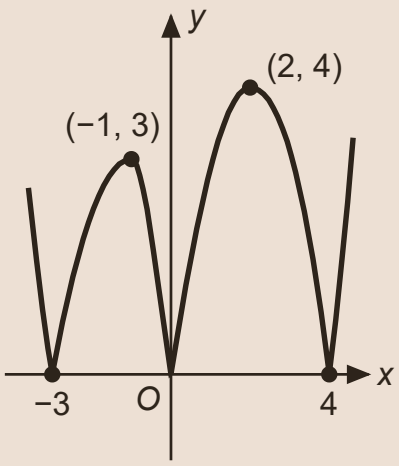
4 marks **Grade B**

1. For $|f(x)|$: reflect the parts below the x-axis; minimum points below the axis become maximum points.
2. For $f(|x|)$: keep the right-hand half and reflect it in the y-axis.
3. Label the new turning points and intercepts.

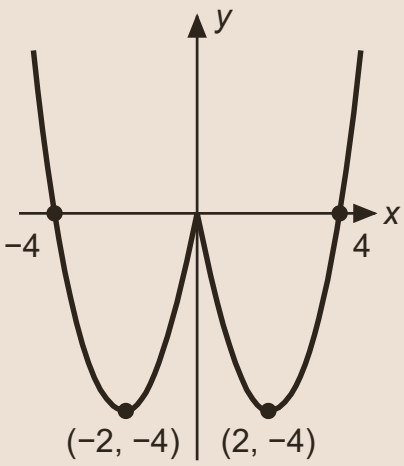
Example. The curve $y = f(x)$ has a maximum at $(-1, 3)$ and a minimum at $(2, -4)$, and crosses the x-axis at $(-3, 0)$, $(0, 0)$ and $(4, 0)$.
Sketch (a) $y = |f(x)|$ (b) $y = f(|x|)$, showing the turning points and the points where each curve meets the x-axis.

Model answer.

(a) y = |f(x)|



(b) y = f(|x|)



(a) No part below the x-axis, meeting it at $(-3, 0)$, $(0, 0)$ and $(4, 0)$ (B1)

Maxima at $(-1, 3)$ and $(2, 4)$ (B1)

(b) Symmetric about the y-axis, crossing at $(-4, 0)$, $(0, 0)$ and $(4, 0)$ (B1)

Minima at $(2, -4)$ and $(-2, -4)$ (B1)

Find a turning point after combined transformations 3 marks **Grade A**

1. Apply the horizontal change to the x-coordinate.
2. Apply the vertical change to the y-coordinate (stretch before translate).
3. A negative multiplier outside f turns a minimum into a maximum.

Example. The curve $y = f(x)$ has a minimum point at $(-2, 5)$. Find the coordinates of the turning point of each curve, and state whether it is a maximum or a minimum:
(a) $y = f(2x + 4)$ (b) $y = 3 - 2f(x)$

Model answer.

(a) $2x + 4 = -2$ gives $x = -3$: minimum at $(-3, 5)$ (B1)

(b) $3 - 2(5) = -7$: the point is $(-2, -7)$ (B1)

and it is a maximum (B1)

Don't lose marks

- Moving the wrong way: $f(x + 3)$ is 3 units left, not right.
- Using scale factor 2 for $f(2x)$ instead of $\frac{1}{2}$
- Forgetting to transform the asymptotes, or not labelling them.
- For $f(|x|)$, reflecting the left-hand part instead of the right-hand part.
- Vague descriptions such as 'moves it' or 'makes it bigger' instead of naming the transformation with its details.

More tips

Memory tricks

- Inside the bracket: do the opposite and divide. Outside: do what it says and multiply.
- To find where a feature moves under $f(x - 3)$, solve $x - 3 =$ its old x-coordinate.
- Vertical asymptotes only move under changes to x; horizontal asymptotes only under changes to y
- A negative multiplier outside f swaps maxima and minima.

Exam technique

- 'Describe fully' needs the type of transformation and every detail (vector; direction and scale factor; or mirror line).
- Label every point the question mentions with its new coordinates, and every asymptote with its equation.
- For combinations, write the image of one key point step by step before you sketch.

What each grade needs

What you need to be able to do, from the first marks up to the top grade.

Grade E

Translate a graph vertically and horizontally

$f(x) + a$ moves the graph up by a; $f(x + a)$ moves it left by a

Grade D

Stretch a graph parallel to either axis

$af(x)$: stretch parallel to the y-axis, scale factor a; $f(ax)$: parallel to the x-axis, scale factor $\frac{1}{a}$

Grade D

Reflect a graph in either axis

$-f(x)$ reflects the graph in the x-axis; $f(-x)$ reflects it in the y-axis.

Grade C

Find where points and asymptotes move

Apply the transformation to coordinates, e.g. under $y = 2f(x)$, $(3, 4)$ moves to $(3, 8)$

Grade C

Describe a single transformation fully

Name it and give every detail: the vector, or the direction and scale factor, or the mirror line.

Grade B

Sketch |f(x)| and f(|x|) from y = f(x)

$|f(x)|$ reflects parts below the x-axis upwards; $f(|x|)$ reflects the $x \geq 0$ part in the y-axis.

Grade A

Apply combinations of transformations in order

e.g. $y = 2f(x - 1) + 3$: move right 1, stretch vertically by factor 2, then move up 3.

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